Martin-Löf Random Brownian Motion

Kelty Allen

University of California, Berkeley

Work is joint with Laurent Bienvenu and Ted Slaman

16 May 2013

Brownian motion as a random function

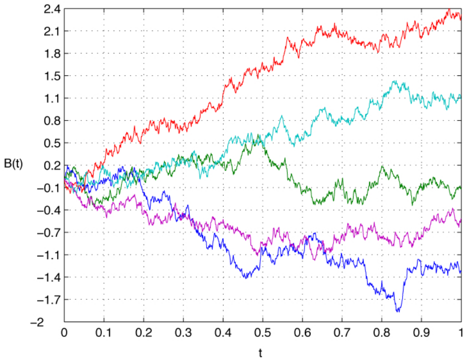

Fig. 1.1. Graphs of five sampled Brownian motions

Figure: Image from "Brownian Motion" by Mörters and Peres

Why Martin-Löf random Brownian Motion?

- Interesting random object
- More insight into "almost surely" results in classical theory of Brownian motion
- More insight into power of algorithmic randomness
- Recursion theoretic proofs of classical results

Introduction

- Asarin and Prokovsky (1986): Complex Oscillations
- Willem Fouché (2000's): Strong foundations; proved Complex Oscillations are the Martin-Löf random sample paths of Brownian motion
- Laurent Bienvenu (2012): Use layerwise computability framework to do further work

Technical Background

Classical Probability Theory

Definition (Classical Brownian Motion)

A collection of functions indexed by time ("real-valued stochastic process"), $\{B(t) : t \in I\}$, is called *standard Brownian motion* iff:

- Initial Value: $\mathrm{B}(0) = 0$,
- Distribution: For all $t \geq 0$ and $h > 0$, the increments $B(t+h) - B(t)$ are normally distributed with expectation 0 and variance h,
- Independent Increments: For all times $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$, the increments $B(t_n) - B(t_{n-1}), B(t_{n-1}) - B(t_{n-2}), \ldots B(t_2) - B(t_1)$ are independent random variables,
- Continuity: Almost surely, the function $t \mapsto B(t)$ is continuous.

These requirements induce a measure on a function space, called *Wiener Measure*. We will talk about Martin-Löf randomness with respect to Wiener Measure on $C[0,1]$.

A little more technical detail

In particular, if $A_1, ..., A_n$ are Borel sets in $\mathbb{R}$, then the probability of the finitary event $[B(t_j) \in A_j \text{ for } 1 \leq j \leq n]$ is given by

$$\mathbb{P}[B(t_j) \in A_j \text{ for } 1 \leq j \leq n] =$$

$$\int_{A_1} \cdots \int_{A_n} \prod \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} e^{\frac{-(y_j - y_{j-1})^2}{2(t_j - t_{j-1})}} dy_n ... dy_1$$

Franklin-Wiener Series Construction of Brownian Motion

We can construct Brownian motion from elements of Cantor space in the following way:
Each function will be represented by an infinite series of the form

$$B(t) = \xi_0 \Delta_0(t) + \xi_1 \Delta_1(t) + \sum_i \sum_{j<2^i} \xi_{i,j} \Delta_{i,j}(t)$$

where the $\xi_{i,j}$ are real-number weights and the $\Delta_{i,j}(t)$ are sawtooth functions.

Franklin-Wiener Series Construction

- Using one binary real α, we split into infinitely many binary reals $\beta_0, \beta_1, \{\beta_{i,j}\}_{i\in\omega,\ j<2^i}$
- We use a Gaussian distribution to map $2^\omega \to [-\infty, \infty]$ and let $\xi_{i,j} = g(\beta_{i,j})$

$$\beta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{g(\beta)} e^{-t^2/2} dt$$

- See picture: Sawtooth functions $\Delta_0, \Delta_1, \Delta_{i,j}$

Franklin-Wiener Series Construction

$$B(t) = \xi_0 \Delta_0(t) + \xi_1 \Delta_1(t) + \sum_i \sum_{j<2^i} \xi_{i,j} \Delta_{i,j}(t)$$

Theorem (Classical Probability Theory - due to Kahane?)

The Franklin-Wiener series representation converges almost surely to a continuous function in $C[0,1]$, and the resulting class of functions is Brownian Motion.

Theorem (Fouché)

The Franklin-Wiener series representation converges to a continuous function for every $\alpha \in 2^\omega$ that is Martin-Löf random.

Layerwise computability

Definition (Layerwise Computable)

For U_n a universal Martin-Löf test on a space (X, μ), let $K_n := X \setminus U_n$. A function $T : (X, \mu) \to Y$ is *layerwise computable* if it is computable on every K_n, uniformly on n.

Theorem (Hoyrup, Rojas)

If $T : (X, \mu) \to Y$ is a layerwise computable map from a computable probability space to a computable metric space, then:

- *The push-forward measure $\nu := \mu \circ T^{-1} \in \mathcal{M}(Y)$ is computable.*
- *T preserves Martin-Löf randomness; i.e. $T(ML_\mu) \subset ML_\nu$. Moreover, there is a constant c (computable from a description of T) such that $T(K_n) \subset K'_{n+c}$ for all n, where K'_{n+c} is the canonical layering of (Y, ν).*

Layerwise computability and MLR Brownian motion

Theorem (Fouché)

The Martin-Löf random paths of Brownian Motion are the image of Martin-Löf random reals in Cantor space under the Franklin-Wiener construction of Brownian Motion.

Theorem (Fouché)

Let $B(t)$ be the Martin-Löf random Brownian motion constructed via the Franklin-Wiener series from a Martin-Löf random real α. There is a piecewise linear function p_m, constructed from the first m bits of α, such that

$$||B - p_m|| < \frac{\log m}{\sqrt{m}}$$

for all $m > M_\alpha$. Moreover, M_α is layerwise computable from α

Useful Properties of Brownian Motion

Classical Modulus of Continuity Results

Theorem (Levy)

For every constant $c < \sqrt{2}$, *almost surely, for every* $\varepsilon > 0$ *there exist* $0 < h < \varepsilon$ *and* $t \in [0, 1-h]$ *with*

$$|B(t+h) - B(t)| \geq c\sqrt{h \log(1/h)}$$

Theorem (Levy)

For every constant $C > \sqrt{2}$, *almost surely, for every sufficiently small* $h > 0$ *and all* $0 < t < 1-h$,

$$|B(t+h) - B(t)| \leq C\sqrt{h \log(1/h)}$$

Modulus of Continuity of MLR Brownian motion

Theorem (Sharpening a result of Fouché)

Let $B(t)$ be a Martin-Löf random Brownian motion. Then for all $c < \sqrt{2}$, for all h_0, there exists $h < h_0$ such that for all t,

$$|B(t+h) - B(t)| \geq c\sqrt{h \log(1/h)}$$

Theorem (Sharpening a result of Fouché)

Let $B(t)$ be a Martin-Löf random Brownian motion. Then for all $C > \sqrt{2}$, there is an h_0, such that for all $h < h_0$ and all t,

$$|B(t+h) - B(t)| \leq C\sqrt{h \log(1/h)}$$

and moreover, h_0 is layerwise computable from B.

Zeros of Brownian Motion

Zeros of Brownian motion

Definition

For a path $B(t)$ of one-dimensional Brownian motion, define $Z_B = \{t \geq 0 : B(t) = 0\}$ to be the zero set of B

Theorem (Classical Probability Theory)

Almost surely, Z_B is a closed set with no isolated points.

Theorem

For $B(t)$ a one-dimensional Martin-Löf random Brownian motion, Z_B is a closed set with no isolated points.

Characterize zeros of Martin-Löf Brownian motion

Necessary:

Theorem

If $\alpha \in (0,1]$ has effective Hausdorff dimension $< 1/2$, $\alpha \notin Z_B$ for any Martin-Löf random Brownian motion B.

Sufficient:

Theorem

If $\alpha \in (0,1]$ has effective Hausdorff dimension $> 1/2$, then there exists a Martin-Löf random Brownian motion B such that $\alpha \in Z_B$.

Some points with effective dimension $1/2$ are zeros, and some are not.

Theorem

For B a Martin-Löf random Brownian path, the first zero of B after any given computable real q is layerwise computable in B.

Proof:

Lemma (1)

It is layerwise computable in $B(t)$ to see that there is a zero in a given interval $[l, r]$ with computable endpoints.

Lemma (2)

It is layerwise computable in $B(t)$ to see that there is not a zero in a given interval $[l, r]$ with computable endpoints.

Proof of Lemma 1 - Demonstrating existence of a zero

- Main Idea: We can see, layerwise computably in B, positive and negative values in $[l, r]$.
- (Fouché): The local minima (and maxima) of Martin-Löf random Brownian motion have non-recursive values.
- Thus 0 is never a local minimum or maximum
- So if a zero exists in $[l, r]$ it is accompanied by intervals, arbitrarily close on either side, where $B(t) > 0$ and $B(t) < 0$.

Proof of Lemma 1 - Demonstrating existence of a zero

- How do we determine, layerwise computably, if $B(s) > 0$ or $B(s) < 0$ for s computable?
- Recall (Fouché): From the first m bits of $\alpha \in 2^\omega$, we can compute a piecewise linear function $p_m(t)$ such that

$$\|B(t) - p_m(t)\| \leq \frac{\log m}{\sqrt{m}}$$

  for all $m > M_\alpha$
- For a fixed s, we compute $p_m(s)$ until $\|p_m(s)\| \geq \frac{\log m}{\sqrt{m}}$ and $m > M_\alpha$, which must eventually happen as $B(s) \neq 0$ and $\frac{\log m}{\sqrt{m}} \to 0$
- By running this computation on any dense set of computable reals, we must eventually observe positive and negative values for $p_m(s)$ which are far enough from 0 that we know $B(t)$ has crossed 0 in $[l, r]$.

Proof of Lemma 2 - Seeing that there is not a zero

- Main Idea: We want to show, layerwise computably, that $B(t)$ is bounded away from 0 on $[l, r]$
- Recall: $\forall c > \sqrt{2}\ \exists h_0$ (layerwise computable in B) s.t. $\forall h < h_0$

$$\|B(t+h) - B(t)\| < c\sqrt{h \log 1/h}$$

- So by sampling $B(t)$ at intervals of size $1/2^{n_i}$, n_i increasing and $1/2^{n_0} < h_0$, we eventually must find $h^* = 1/2^{n_i}$ such that

$$B(t) > 2\sqrt{h^* \log 1/h^*}$$

for all $t \in \{l + k/2^n\}_{k<2^n}$ in $[l, r]$.

Completion of the proof

- Find the first zero of a Brownian path $B(t)$ after a given computable real q
- Divide interval $[q, 1]$ into intervals of size $1/2^{n_0}$ for a suitable n_0
- Run algorithms from lemmas to find the closest interval $[l_0, r_0]$ to q that contains a zero.
- Divide $[l_0, r_0]$ into intervals of size $1/2^{n_1}$ and repeat
- Layerwise computably in B, we find sequences $\{l_i\}$ and $\{r_i\}$ which converge to the first zero after q from the left and from the right.
- Note: We could run a similar argument for crossing any computable real value after a computable time.

An application to the Dirichlet Problem

Brownian motion in other spaces

Brownian motion can be defined on $C[0, \infty)$, and can be constructed in a similar way by "gluing together" Brownian motion paths defined on $[0, 1]$

Definition

$B(t) = (B_1(t), B_2(t), ..., B_d(t))$, $t \geq 0$, is *d-dimensional (standard) Brownian motion* if $B_1, ..., B_d(t)$ are independent standard 1-dimensional Brownian motions.

Theorem (Fouché)

$B(t) = (B_1(t), B_2(t), ..., B_d(t))$, $t \geq 0$, is d-dimensional Martin-Löf random Brownian motion if $B_1, ..., B_d(t)$ are mutually Martin-Löf random 1-dimensional Brownian motions.

Dirichlet Problem

Definition (Dirichlet Problem)

Given a function ϕ that has values everywhere on the boundary of a region in $\mathbb{R}^d$, is there a unique continuous function u twice continuously differentiable in the interior and continuous on the boundary such that u is harmonic in the interior and $u = \phi$ on the boundary?

Dirichlet Problem

Theorem (Kakutani)

Suppose $U \subset \mathbb{R}^d$ *is a bounded domain such that every boundary point satisfies the Poincaré cone condition, and suppose* ϕ *is a continuous function on the boundary* ∂U*. Let* $\tau(\partial U) = \inf\{t > 0 : B(t) \in \partial U\}$*, which is an almost surely finite stopping time. Then the function* $u : \overline{U} \to \mathbb{R}$ *given by*

$$u(x) = \mathbb{E}_x \left[\phi(B(\tau(\partial U)))\right], \qquad \text{for } x \in \overline{U},$$

is the unique continuous function harmonic on U *with* $u(x) = \phi(x)$ *for all* $x \in \partial U$*.*

Computable Dirichlet Problem

Strong Conjecture

Let U be a bounded domain with computable boundary ∂U. Let ϕ be a computable continuous function on ∂U. Then there is a unique, continuous, computable function $u : \overline{U} \to \mathbb{R}$ harmonic on U such that $u(x) = \phi(x)$ for all $x \in \partial U$.

Further Directions

- Other types of randomness for Wiener measure
- Many "almost surely" results in classical theory
  - How much randomness do they require?
  - Do they characterize familiar notions of randomness?
- Can this more constructive approach be used to solve open problems in the classical theory?

Thank You!