Higher randomness and triviality

Laurent Bienvenu (LIAFA, CNRS & Université Paris 7)

Noam Greenberg (Victoria University of Wellington)

Benoît Monin (LIAFA, CNRS & Université Paris 7)

Thematic Semester on Computability and Randomness

Buenos Aires 2013

1. Higher randomness: the basics

Δ^1_1 randomness

In his seminal work on randomness, Martin-Löf did not only define the notion we now know as Martin-Löf randomness.

Δ^1_1 randomness

In his seminal work on randomness, Martin-Löf did not only define the notion we now know as Martin-Löf randomness.

Indeed, he also introduced a higher notion of randomness: Δ^1_1-randomness, where X is Δ^1_1 random if and only if it does not belong to any Δ^1_1 set of measure 0.

Δ^1_1-ML-randomness

Another way to define randomness via Δ^1_1 objects would be to mimic Martin-Löf's definition for the classical case:

Definition

A Δ^1_1-Martin-Löf test is a sequence $(\mathcal{U}_n)$ of uniformly Δ^1_1 open sets such that $\mu(\mathcal{U}_n) \leq 2^{-n}$. A sequence X is Δ^1_1-Martin-Löf random if $X \notin \bigcap_n \mathcal{U}_n$ for all Δ^1_1-Martin-Löf tests.

Δ^1_1-ML-randomness

Another way to define randomness via Δ^1_1 objects would be to mimic Martin-Löf's definition for the classical case:

Definition

A Δ^1_1-Martin-Löf test is a sequence $(\mathcal{U}_n)$ of uniformly Δ^1_1 open sets such that $\mu(\mathcal{U}_n) \leq 2^{-n}$. A sequence X is Δ^1_1-Martin-Löf random if $X \notin \bigcap_n \mathcal{U}_n$ for all Δ^1_1-Martin-Löf tests.

As it turns out, these two concepts are equivalent:

Theorem (Sacks)

Δ^1_1-randomness and Δ^1_1-Martin-Löf randomness coincide.

Π^1_1/Σ^1_1 versions of randomness

Of course we can do the same with Σ^1_1 and Π^1_1 instead of Δ^1_1.

Π^1_1/Σ^1_1 versions of randomness

Of course we can do the same with Σ^1_1 and Π^1_1 instead of Δ^1_1.

- For Σ^1_1, everything collapses (Sacks):

$$\Sigma^1_1\text{-random} = \Sigma^1_1\text{-ML random} = \Delta^1_1\text{-random}$$

Π^1_1/Σ^1_1 versions of randomness

Of course we can do the same with Σ^1_1 and Π^1_1 instead of Δ^1_1.

- For Σ^1_1, everything collapses (Sacks):

$$\Sigma^1_1\text{-random} = \Sigma^1_1\text{-ML random} = \Delta^1_1\text{-random}$$

- The situation is a lot more interesting for Π^1_1 randomness...

Π^1_1 randomness

Definition
X is Π^1_1-random if it does not belong to any Π^1_1 nullset.

Π^1_1 randomness

Definition

X is Π^1_1-random if it does not belong to any Π^1_1 nullset.

Theorem (Kechris)

There exists a Π^1_1 nullset which contains all others.

Π^1_1 randomness

Definition

X is Π^1_1-random if it does not belong to any Π^1_1 nullset.

Theorem (Kechris)

There exists a Π^1_1 nullset which contains all others.

Thus a sequence X is Π^1_1-random iff it avoids this maximal Π^1_1-nullset.

Π^1_1 ML-randomness

Definition

A Π^1_1 Martin-Löf test is a sequence $(\mathcal{U}_n)$ of uniformly Π^1_1 open sets such that $\mu(\mathcal{U}_n) \leq 2^{-n}$. A sequence X is Π^1_1-Martin-Löf random if $X \notin \bigcap_n \mathcal{U}_n$ for all Π^1_1-Martin-Löf tests.

Π^1_1 ML-randomness

Definition

A Π^1_1 Martin-Löf test is a sequence $(\mathcal{U}_n)$ of uniformly Π^1_1 open sets such that $\mu(\mathcal{U}_n) \leq 2^{-n}$. A sequence X is Π^1_1-Martin-Löf random if $X \notin \bigcap_n \mathcal{U}_n$ for all Π^1_1-Martin-Löf tests.

It is easy to see that if $(\mathcal{U}_n)$ is a Π^1_1-ML test, then $\bigcap_n \mathcal{U}_n$ is a Π^1_1 set. Thus Π^1_1-randomness implies Π^1_1-ML randomness.

A digression

In higher recursion theory, the analog of the class of c.e. sets of integers (or strings, or...) is the class of Π^1_1 sets of integers. Indeed, by the Gandy-Spector theorem, one can think of a Π^1_1 set of integers as being given by an enumeration with stages $\{s \mid s < \omega_1^{ck}\}$.

A digression

In higher recursion theory, the analog of the class of c.e. sets of integers (or strings, or...) is the class of Π^1_1 sets of integers. Indeed, by the Gandy-Spector theorem, one can think of a Π^1_1 set of integers as being given by an enumeration with stages $\{s \mid s < \omega_1^{ck}\}$.

Bottom setting	Higher analogue
c.e.	Π^1_1
finite c.e.	Δ^1_1
computable	Δ^1_1
$\emptyset'$	$\mathcal{O}$
$\leq_T$	???

Thus, Π^1_1 ML-randomness really is the analogue of Martin-Löf randomness in the higher setting.

A digression

This is a powerful analogy, and so hereafter we'll refer to Π^1_1 and Δ^1_1 sets of integers as **c.e.** and **computable**, respectively.

A digression

This is a powerful analogy, and so hereafter we'll refer to Π^1_1 and Δ^1_1 sets of integers as **c.e.** and **computable**, respectively.

For example, we can state: it is possible to **enumerate** all Π^1_1-Martin-Löf tests (or **Martin-Löf tests!**) and thus there exists a universal one.

Δ^1_1-randomness vs Π^1_1-ML randomness

With this correspondence in mind, we see that Δ^1_1-randomness is defined in terms of **computable/finite** objects, thus it should be related to **computable** randomness or **Schnorr** randomness....

Δ^1_1-randomness vs Π^1_1-ML randomness

With this correspondence in mind, we see that Δ^1_1-randomness is defined in terms of **computable/finite** objects, thus it should be related to **computable** randomness or **Schnorr** randomness....

... and indeed it coincides with both.

Δ^1_1-randomness vs Π^1_1-ML randomness

With this correspondence in mind, we see that Δ^1_1-randomness is defined in terms of **computable/finite** objects, thus it should be related to **computable** randomness or **Schnorr** randomness....

... and indeed it coincides with both.

Using the usual techniques, one can thus show that Π^1_1-ML randomness is strictly stronger than Δ^1_1-randomness.

Π^1_1-randomness vs Π^1_1-ML randomness

Π^1_1-randomness and Π^1_1-Martin-Löf randomness are related by the following beautiful theorem:

Theorem (Hjorth-Nies / Chong-Nies-Yu)

The following are equivalent:

Π^1_1-randomness vs Π^1_1-ML randomness

Π^1_1-randomness and Π^1_1-Martin-Löf randomness are related by the following beautiful theorem:

Theorem (Hjorth-Nies / Chong-Nies-Yu)

The following are equivalent:

- *X is Π^1_1 random*
- *X is Π^1_1-ML random and $\omega_1^X = \omega_1^{ck}$.*
- *X is Δ^1_1-ML random and $\omega_1^X = \omega_1^{ck}$.*

Π^1_1-randomness vs Π^1_1-ML randomness

Π^1_1-randomness and Π^1_1-Martin-Löf randomness are related by the following beautiful theorem:

Theorem (Hjorth-Nies / Chong-Nies-Yu)

The following are equivalent:

- *X is Π^1_1 random*
- *X is Π^1_1-ML random and $\omega_1^X = \omega_1^{ck}$.*
- *X is Δ^1_1-ML random and $\omega_1^X = \omega_1^{ck}$.*

And thus it suffices to separate the first two notions to show that there exists a Π^1_1-ML-random such that $\omega_1^X > \omega_1^{ck}$.

Π^1_1-randomness vs Π^1_1-ML randomness

Π^1_1-randomness and Π^1_1-Martin-Löf randomness are related by the following beautiful theorem:

Theorem (Hjorth-Nies / Chong-Nies-Yu)

The following are equivalent:

- *X is Π^1_1 random*
- *X is Π^1_1-ML random and $\omega_1^X = \omega_1^{ck}$.*
- *X is Δ^1_1-ML random and $\omega_1^X = \omega_1^{ck}$.*

And thus it suffices to separate the first two notions to show that there exists a Π^1_1-ML-random such that $\omega_1^X > \omega_1^{ck}$. In fact, Ω is such a real (although $\Omega \not\geq_T \mathcal{O}$).

Summing up

$$\Pi^1_1\text{-RAND}$$
$$\Downarrow$$
$$\Pi^1_1\text{-MLR}$$
$$\Downarrow$$
$$\Delta^1_1\text{-Rand} = \Delta^1_1\text{-MLR} = \Sigma^1_1\text{-Rand} = \Sigma^1_1\text{-MLR}$$

And the implications are strict.

weak-2-randomness

With the same intuition, the analog of weak-2-randomness is the following

Definition

A **weak-2-test** is a sequence $(\mathcal{U}_n)$ of uniformly **c.e.** open sets such that $\mu(\mathcal{U}_n) \longrightarrow 0$. A sequence X is **weak-2-random** if $X \notin \bigcap_n \mathcal{U}_n$ for all **weak-2-tests**.

weak-2-randomness

One can easily see: Π^1_1-RAND $\Rightarrow$ **weak-2-RAND** $\Rightarrow$ Π^1_1-MLR

weak-2-randomness

One can easily see: Π^1_1-RAND $\Rightarrow$ **weak-2-RAND** $\Rightarrow$ Π^1_1-MLR

The second implication is strict, as we will see.

weak-2-randomness

One can easily see: Π^1_1-RAND $\Rightarrow$ **weak-2-RAND** $\Rightarrow$ Π^1_1-MLR

The second implication is strict, as we will see.

A seemingly difficult open question: are **weak-2-randomness** and Π^1_1-randomness equivalent?

Kolmogorov complexity

Just like in the classical setting, one can define **Kolmogorov complexity**, denoted K, via a universal prefix-free **machine**, or via a universal **left-c.e.** semimeasure.

Kolmogorov complexity

Just like in the classical setting, one can define **Kolmogorov complexity**, denoted K, via a universal prefix-free **machine**, or via a universal **left-c.e.** semimeasure.

The Levin-Schnorr theorem still holds: X is **ML random** if and only if $\mathrm{K}\left(X \upharpoonright n\right) \geq n - O(1)$.

2. Higher Turing reductions

Higher reductions

If we want to investigate the relations between randomness and Turing degrees, we need to first find the correct analog of Turing reduction.

Higher reductions

If we want to investigate the relations between randomness and Turing degrees, we need to first find the correct analog of Turing reduction.

What does it mean for X be to be **computable in** Y?

Higher reductions

If we want to investigate the relations between randomness and Turing degrees, we need to first find the correct analog of Turing reduction.

What does it mean for X be to be **computable in** Y?

The most obvious definition: X is Δ^1_1 in A, meaning that there is a Π^1_1-set P and a Σ^1_1 set S such that

$$n \in X \Leftrightarrow (A, n) \in P \Leftrightarrow (A, n) \in S$$

Higher reductions

If we want to investigate the relations between randomness and Turing degrees, we need to first find the correct analog of Turing reduction.

What does it mean for X be to be **computable in** Y?

The most obvious definition: X is Δ^1_1 in A, meaning that there is a Π^1_1-set P and a Σ^1_1 set S such that

$$n \in X \Leftrightarrow (A, n) \in P \Leftrightarrow (A, n) \in S$$

We also write $X \leq_h A$ as by a result of Kleene, this is equivalent for X to be hyperarithmetic in A (i.e. $X \leq_T A^{(\gamma)}$ for some $\gamma < \omega_1^A$).

Higher reductions

The problem with this reduction is of topological nature. When studying the interactions between randomness and Turing degrees, we make great use of the fact that Turing reductions are continuous, i.e., that when we have $\Phi^{Y} = X$, only a finite prefix of Y is used in the computation of a given prefix of X.

Higher reductions

The problem with this reduction is of topological nature. When studying the interactions between randomness and Turing degrees, we make great use of the fact that Turing reductions are continuous, i.e., that when we have $\Phi^Y = X$, only a finite prefix of Y is used in the computation of a given prefix of X.

The solution: in the classical setting, a Turing reduction Φ can be viewed as a c.e. set of pairs of strings

Higher reductions

Definition (BGM)

A **Turing reduction** Φ is a **c.e.** set of pairs of strings. We write $\Phi^Y = X$ if

- $(\forall n)\ (\exists k)\ \langle Y \upharpoonright k, X \upharpoonright n \rangle \in \Phi$
- $(\forall k)\ \left[(\exists \tau)\ \langle Y \upharpoonright k, \tau \rangle \in \Phi \Rightarrow \tau \preceq X \right]$

Higher reductions

Definition (BGM)

A **Turing reduction** Φ is a **c.e.** set of pairs of strings. We write $\Phi^Y = X$ if

- $(\forall n)\ (\exists k)\ \langle Y \upharpoonright k, X \upharpoonright n \rangle \in \Phi$
- $(\forall k)\ \left[(\exists \tau)\ \langle Y \upharpoonright k, \tau \rangle \in \Phi \Rightarrow \tau \preceq X \right]$

We **do not** require the domain of Φ to be closed under prefixes, as Hjorth and Nies do with the notion of fin-h reducibility.

Higher reductions

We (of course) denote higher Turing reducibility by $\leq_T$.

Higher reductions

We (of course) denote higher Turing reducibility by $\leq_T$.

Although this may not be clear from the definition, $\leq_T$ is a refinement of $\leq_h$:

$$X \leq_T Y \Rightarrow X \leq_h Y$$

First results... for free!

With this translation of “computable” to the higher setting, we get many theorems for free by a **direct** adaptation of the classical setting.

First results... for free!

With this translation of “computable” to the higher setting, we get many theorems for free by a **direct** adaptation of the classical setting.

Theorem (Kučera-Gács / Hjorth-Nies)

- *For every X, there exists Y which is* Π^1_1*-ML random and* $X \leq_T Y$ *(and thus* $X \leq_h Y$*).*
- *For every* $X \geq_T \mathcal{O}$*, there exists a* Π^1_1*-ML random Y such that* $X \equiv_T Y$

First results... for free!

With this translation of “computable” to the higher setting, we get many theorems for free by a **direct** adaptation of the classical setting.

Theorem (Kučera-Gács / Hjorth-Nies)

- *For every X, there exists Y which is Π^1_1-ML random and $X \leq_T Y$ (and thus $X \leq_h Y$).*
- *For every $X \geq_T \mathcal{O}$, there exists a Π^1_1-ML random Y such that $X \equiv_T Y$*

Theorem (Chaitin, adapted)

*There exists a **left-c.e.** Π^1_1-Martin-Löf random real; call it Ω. Any such real **computes** $\mathcal{O}$.*

Relativization

In this context, we can define c.e. operators W_e and an A**-c.e.** set S of integers is any set of the form W_e^A.

Relativization

In this context, we can define c.e. operators W_e and an A-**c.e.** set S of integers is any set of the form W_e^A.

One can similarly define A-**c.e.** open set as an open set generated by a A-**c.e.** set of strings and thus define A-**MLR**.

Relativization

In this context, we can define c.e. operators W_e and an ***A*-c.e.** set S of integers is any set of the form W_e^A.

One can similarly define A-**c.e.** open set as an open set generated by a A-**c.e.** set of strings and thus define A-**MLR**.

Theorem (van Lambalgen, adapted)

*$X \oplus Y$ is **Martin-Löf random** if and only if X is **Martin-Löf random** and Y is X-**Martin-Löf random**.*

What does NOT work in the higher case?

Some classical techniques of the classical case do not translate in the higher case.

What does NOT work in the higher case?

Some classical techniques of the classical case do not translate in the higher case.

- It is not true any more that for a **c.e.** open set $\mathcal{U}$ and cylinder $[\sigma]$, the set $[\sigma] \setminus \mathcal{U}[s]$ clopen, because $\mathcal{U}[s]$ may no longer be clopen for $s \geq \omega$.

What does NOT work in the higher case?

Some classical techniques of the classical case do not translate in the higher case.

- It is not true any more that for a **c.e.** open set $\mathcal{U}$ and cylinder $[\sigma]$, the set $[\sigma] \setminus \mathcal{U}[s]$ clopen, because $\mathcal{U}[s]$ may no longer be clopen for $s \geq \omega$.
- "Time tricks" don't work. There are several theorems in algorithmic randomness that exploit the length of a string as an enumeration time (e.g. $\mathrm{K}^{\sigma}_{|\sigma|}$).

2-MLR and $\mathcal{O}$-**MLR**

A well-known theorem of Miller/Nies-Stephan-Terwijn: 2-MLR reals are characterized by the condition:

$$\mathrm{C}(X \upharpoonright n) \geq n - c \text{ for some } c \text{ and infinitely many } n \quad (\diamond)$$

The proof of $(\diamond) \Rightarrow$ 2-MLR involves a time trick. This does not translate...

2-MLR and $\mathcal{O}$-**MLR**

A well-known theorem of Miller/Nies-Stephan-Terwijn: 2-MLR reals are characterized by the condition:

$$\mathrm{C}(X \upharpoonright n) \geq n - c \text{ for some } c \text{ and infinitely many } n \quad (\diamond)$$

The proof of $(\diamond) \Rightarrow$ 2-MLR involves a time trick. This does not translate... and indeed, the condition

$$\mathrm{C}(X \upharpoonright n) \geq n - c \text{ for some } c \text{ and infinitely many } n \quad (\diamond\diamond)$$

is **not** equivalent to being $\mathcal{O}$-**MLR**; it is somewhere between **MLR** and Π^1_1-random [A recent result of Greenberg and Slaman: $\mathcal{O}$-**ML randomness** implies (strictly) Π^1_1-randomness].

Returning to the class **W2R**

Another result which does not (directly) translate:

Returning to the class **W2R**

Another result which does not (directly) translate:

Theorem (Hirschfeldt-Miller)

A real is weakly-2-random if and only if it is Martin-Löf random and forms a minimal pair with $\emptyset'$.

Returning to the class **W2R**

Another result which does not (directly) translate:

Theorem (Hirschfeldt-Miller)

A real is weakly-2-random if and only if it is Martin-Löf random and forms a minimal pair with $\emptyset'$.

It is **not** the case that a set is **weakly-2-random** iff it forms a (**Turing**) minimal pair with $\mathcal{O}$ [indeed, by the Gandy basis theorem, there exists an $\mathcal{O}$-computable Π^1_1-random, hence **weakly-2-random**].

W2R and Π^1_1-RAND: separation attempts

Π^1_1-randoms are **Martin-Löf randoms** X such that $\omega_1^X = \omega_1^{ck}$. This is equivalent to $X \not\geq_h \mathcal{O}$.

W2R and Π^1_1-RAND: separation attempts

Π^1_1-randoms are **Martin-Löf randoms** X such that $\omega_1^X = \omega_1^{ck}$. This is equivalent to $X \not\geq_h \mathcal{O}$.

So asking whether **weak-2-randomness** equals Π^1_1-randomness amounts to asking whether there is a **weak-2-random** X such that $X \geq_h \mathcal{O}$.

W2R and Π^1_1-RAND: separation attempts

Π^1_1-randoms are **Martin-Löf randoms** X such that $\omega_1^X = \omega_1^{ck}$. This is equivalent to $X \not\geq_h \mathcal{O}$.

So asking whether **weak-2-randomness** equals Π^1_1-randomness amounts to asking whether there is a **weak-2-random** X such that $X \geq_h \mathcal{O}$.

An weaker question: is there a **weak-2-random** X such that $X \geq_T \mathcal{O}$?

W2R and Π^1_1-RAND: separation attempts

Π^1_1-randoms are **Martin-Löf randoms** X such that $\omega_1^X = \omega_1^{ck}$. This is equivalent to $X \not\geq_h \mathcal{O}$.

So asking whether **weak-2-randomness** equals Π^1_1-randomness amounts to asking whether there is a **weak-2-random** X such that $X \geq_h \mathcal{O}$.

An weaker question: is there a **weak-2-random** X such that $X \geq_T \mathcal{O}$?

Proposition (BGM)

*No **weak-2-random** X is such that $X \geq_T \mathcal{O}$.*

W2R and Π^1_1-RAND: separation attempts

Ω is **ML-random**. Can we get more?

W2R and Π^1_1-RAND: separation attempts

Ω is **ML-random**. Can we get more?

Theorem (Yu)

*Ω is not **weak-2-random**.*

W2R and Π^1_1-RAND: separation attempts

Ω is **ML-random**. Can we get more?

Theorem (Yu)

*Ω is not **weak-2-random**.*

Maybe the halves of Ω?

W2R and Π^1_1-RAND: separation attempts

Ω is **ML-random**. Can we get more?

Theorem (Yu)

Ω *is not* ***weak-2-random****.*

Maybe the halves of Ω?

Theorem (BGM)

Ω_0 *is not* ***weak-2-random****. In fact, no* $X \leq_{tt} \mathcal{O}$ *is.*

A conjecture

One way to separate **weak-2-randomness** from Π^1_1-randomness would be to evaluate the Borel rank of the class of Π^1_1-randomness.

A conjecture

One way to separate **weak-2-randomness** from Π^1_1-randomness would be to evaluate the Borel rank of the class of Π^1_1-randomness.

The Borel rank of the set of **weak-2-randoms** is $\boldsymbol{\Pi}^0_3$. The set of Π^1_1-randoms must have much higher rank, but we don't know how to prove this.

3. Higher lowness and triviality

Complexity and semi-measures

Recall that a (discrete) semimeasure is a function $m : \omega \to \mathbb{R}^+$ such that

$$\sum_i m(i) \leq 1$$

we say that it is (left-)c.e. if its lower graph is c.e.

Proposition

There exists a universal c.e. semimeasure, i.e. a c.e. semimeasure $\tilde{m}$ such that for any other c.e. semimeasure μ one has

$$\tilde{m} >^{\times} \mu$$

Complexity and semi-measures

Recall that a (discrete) semimeasure is a function $m : \omega \to \mathbb{R}^+$ such that

$$\sum_i m(i) \leq 1$$

we say that it is (left-)**c.e.** if its lower graph is **c.e.**

Proposition

There exists a universal **c.e.** *semimeasure, i.e. a* **c.e.** *semimeasure* $\tilde{m}$ *such that for any other* **c.e.** *semimeasure* μ *one has*

$$\tilde{m} >^\times \mu$$

Levin's coding theorem

Theorem (Levin)

Given a left-c.e. semimeasure m, one can uniformly build a prefix-free machine M such that $\mathrm{K}_M = -\log m \pm 1$.

Levin's coding theorem

Theorem (Levin)

Given a left-c.e. semimeasure m, one can uniformly build a prefix-free machine M such that $\mathrm{K}_M = -\log m \pm 1$.

To prove Levin's theorem, one can assume that m is dyadic and whenever a new amount 2^{-k} is added to some i, a new description of i is issued in M. The new description is carefully chosen to allow (potentially short) future descriptions. This is done by keeping track of the binary expansion of $\sum_i m(i)[s]$ at each stage s.

Levin's coding theorem

In the higher setting, a bit more care is needed as the binary expansion of $\sum_i m(i)[s]$ can now be an infinite object, but it turns out that the only case where this could cause an issue is when the binary expansion is of the form

$$0.****011111111111111\ldots$$

which is equal

$$0.****100000000000000\ldots$$

and thus is finite.

Levin's coding theorem

This observation was made by Nies and Hjorth, who thus obtained:

Theorem (Nies-Hjorth)

*Given a **c.e.** semimeasure m, one can uniformly build a prefix-free **machine** M such that $\mathrm{K}_M = -\log m \pm 1$.*

K-triviality

One of the big advances of recent years's research in algorithmic randomness is the discovery of the (properties of) the class of **K-trivial** reals.

K-triviality

One of the big advances of recent years's research in algorithmic randomness is the discovery of the (properties of) the class of **K-trivial** reals.

Definition

A real X is **K-trivial** if

$$\mathrm{K}(A \upharpoonright n) \leq^{+} \mathrm{K}(n)$$

These are the reals that are as far as random as possible from the point of view of K. Computable reals are K-trivial but Solovay was able to construct a non-computable c.e. K-trivial.

K-triviality

A remarkable series of results (many authors with Nies as the least common denominator):

K-triviality

A remarkable series of results (many authors with Nies as the least common denominator):

Theorem

The following are equivalent:

K-triviality

A remarkable series of results (many authors with Nies as the least common denominator):

Theorem

The following are equivalent:

- *A is K-trivial*

K-triviality

A remarkable series of results (many authors with Nies as the least common denominator):

Theorem

The following are equivalent:

- *A is K-trivial*
- *A is low for MLR:* $\mathit{MLR}^A = \mathit{MLR}$

K-triviality

A remarkable series of results (many authors with Nies as the least common denominator):

Theorem

The following are equivalent:

- *A is K-trivial*
- *A is low for MLR:* $\mathit{MLR}^A = \mathit{MLR}$
- *A is low for* K*:* $\mathrm{K}^A =^+ \mathrm{K}$ *(or* $\tilde{m}^A =^\times \tilde{m}$*)*

K-triviality

A remarkable series of results (many authors with Nies as the least common denominator):

Theorem

The following are equivalent:

- *A is K-trivial*
- *A is low for MLR:* $\mathit{MLR}^A = \mathit{MLR}$
- *A is low for* K*:* $\mathrm{K}^A =^+ \mathrm{K}$ *(or* $\tilde{m}^A =^\times \tilde{m}$*)*
- *A is a base for ML randomness: there exists an A-ML random real X such that* $X \geq_T A$*.*

Relativizing Kolmogorov complexity

We would like to adapt this in the higher setting. It looks like everything should work out the same way, except for one tiny detail....

Relativizing Kolmogorov complexity

We would like to adapt this in the higher setting. It looks like everything should work out the same way, except for one tiny detail....

In the classical setting, $\tilde{m}^A$ is defined in terms of a **uniformly universal c.e. semimeasure**. By this, we mean a function

$$\tilde{m}_2 : 2^{<\omega} \times 2^{\omega} \to \mathbb{R}^+$$

- $\tilde{m}_2(.,.)$ is left-c.e. in both arguments, i.e. $\tilde{m}_2(x, A) > q$ is a Σ^0_1-statement
- **For all** A, $\sum_i \tilde{m}_2(x, A) \leq 1$
- **For all** A and every A-c.e. semimeasure μ, $\tilde{m}_2(., A) >^{\times} \mu$

Relativizing Kolmogorov complexity

This is heavily used in the classical theory. Unfortunately(?), this does not hold any more in the higher setting.

Relativizing Kolmogorov complexity

This is heavily used in the classical theory. Unfortunately(?), this does not hold any more in the higher setting.

Theorem (BGM)

There is no uniformly universal ***c.e.*** *semimeasure (although there is an optimal element among uniform semimeasures).*

Relativizing Kolmogorov complexity

This is heavily used in the classical theory. Unfortunately(?), this does not hold any more in the higher setting.

Theorem (BGM)

There is no uniformly universal ***c.e.*** *semimeasure (although there is an optimal element among uniform semimeasures).*

Actually the situation is even worse.

Relativizing Kolmogorov complexity

This is heavily used in the classical theory. Unfortunately(?), this does not hold any more in the higher setting.

Theorem (BGM)

There is no uniformly universal ***c.e.*** *semimeasure (although there is an optimal element among uniform semimeasures).*

Actually the situation is even worse.

Theorem (BGM)

*There exists an A such that there is no A-****c.e.*** *universal semimeasure.*

Relativizing Kolmogorov complexity

So **it does not make sense in general to talk about** K^A for a given A.

Relativizing Kolmogorov complexity

So **it does not make sense in general to talk about** K^A for a given A.

[Note: we showed that it does make sense in some particular cases: when A is **ML random**, when $A \leq_{tt} \mathcal{O}$, when A is **1-generic**....]

Relativizing Kolmogorov complexity

So **it does not make sense in general to talk about** K^A for a given A.

[Note: we showed that it does make sense in some particular cases: when A is **ML random**, when $A \leq_{tt} \mathcal{O}$, when A is **1-generic**....]

So one more quantifier is needed:

Definition

A is **low for** K if for every A-**c.e.** semimeasure μ one has $\tilde{m} >^{\times} \mu$.

K-trivials are well behaved

Theorem

The following are equivalent:

K-trivials are well behaved

Theorem

The following are equivalent:

- *A is* ***K-trivial***

K-trivials are well behaved

Theorem

The following are equivalent:

- *A is **K-trivial***
- *A is **low for MLR**:* $\mathbf{MLR}^A = \mathbf{MLR}$

K-trivials are well behaved

Theorem

The following are equivalent:

- *A is **K-trivial***
- *A is **low for MLR**:* $\mathbf{MLR}^A = \mathbf{MLR}$
- *A is **low for** K*

K-trivials are well behaved

Theorem

The following are equivalent:

- *A is* ***K-trivial***
- *A is* ***low for MLR****:* $\textbf{\textit{MLR}}^A = \textbf{\textit{MLR}}$
- *A is* ***low for*** K
- *A is a* ***base for ML randomness****: there exists an A-****ML random*** *real X such that* $X \geq_T A$.

K-trivials are well behaved

Theorem

The following are equivalent:

- *A is **K-trivial***
- *A is **low for MLR**:* $\textbf{\textit{MLR}}^A = \textbf{\textit{MLR}}$
- *A is **low for** K*
- *A is a **base for ML randomness**: there exists an A-**ML random** real X such that $X \geq_T A$.*

[and there are some **non-computable** ones: Solovay's construction still works]

K-trivials are well behaved

PROOF

$\mathbb{K}$-trivials are well behaved

PROOF

- **K-trivial** implies **low for** $\mathbb{K}$ by the decanter method (there is no obstacle to translation to higher setting).

$\mathbb{K}$-trivials are well behaved

PROOF

- **K-trivial** implies **low for** $\mathbb{K}$ by the decanter method (there is no obstacle to translation to higher setting).
- **low for** $\mathbb{K}$ implies **low for MLR**: if X is not A-**MLR**, it can be compressed by **some** A-**machine**, even though this machine may not be optimal.

K-trivials are well behaved

PROOF

- **K-trivial** implies **low for K** by the decanter method (there is no obstacle to translation to higher setting).
- **low for K** implies **low for MLR**: if X is not A-**MLR**, it can be compressed by **some** A-**machine**, even though this machine may not be optimal.
- **low for MLR** implies **base for ML randomness**: this is a direct consequence of the higher Kučera-Gács theorem.

$\mathbb{K}$-trivials are well behaved

PROOF

- **K-trivial** implies **low for** $\mathbb{K}$ by the decanter method (there is no obstacle to translation to higher setting).
- **low for** $\mathbb{K}$ implies **low for MLR**: if X is not A-**MLR**, it can be compressed by **some** A-**machine**, even though this machine may not be optimal.
- **low for MLR** implies **base for ML randomness**: this is a direct consequence of the higher Kučera-Gács theorem.
- **Base for ML randomness** implies **K-trivial**: this is done via the hungry sets constructions, with some careful adaptations.

Caveat

Caveat: in Nies' book, it is claimed that

$$X \text{ base for } \Pi^1_1\text{-ML-randomness} \iff X \text{ is } \Delta^1_1.$$

Caveat

Caveat: in Nies' book, it is claimed that

$$X \text{ base for } \Pi^1_1\text{-ML-randomness} \quad \Leftrightarrow \quad X \text{ is } \Delta^1_1.$$

This is not a mistake. This is because Nies means something different: he calls A-Martin-Löf random a real which passes all the Martin-Löf tests which are $\Pi^1_1[A]$ (equivalently, Σ_1-definable in $\mathbb{L}_{\omega_1^A}[A]$).

K-trivials are well behaved - bis

Many of the characteristic features of the K-trivials translate to the higher setting.

K-trivials are well behaved - bis

Many of the characteristic features of the K-trivials translate to the higher setting.

- **K-trivials** form a **Turing**-ideal.

K-trivials are well behaved - bis

Many of the characteristic features of the K-trivials translate to the higher setting.

- **K-trivials** form a **Turing**-ideal.
- They are all $\leq_{tt} \mathcal{O}$.

K-trivials are well behaved - bis

Many of the characteristic features of the K-trivials translate to the higher setting.

- **K-trivials** form a **Turing**-ideal.
- They are all $\leq_{tt} \mathcal{O}$.
- Every **K-trivial** is **Turing** below a **c.e. K-trivial**.

K-trivials are well behaved - bis

Many of the characteristic features of the K-trivials translate to the higher setting.

- **K-trivials** form a **Turing**-ideal.
- They are all $\leq_{tt} \mathcal{O}$.
- Every **K-trivial** is **Turing** below a **c.e. K-trivial**.
- **K-trivials** are **jump traceable**.

Open questions

Open questions

Is there an universal A-Martin-Löf test for all A?

Open questions

Is there an universal A-Martin-Löf test for all A?

Separate **weak-2-randomness** from Π^1_1-randomness.

Open questions

Is there an universal A-Martin-Löf test for all A?

Separate **weak-2-randomness** from Π^1_1-randomness.

Characterize the set of reals X such that $\mathrm{C}(X \upharpoonright n) > n - c$ i.o.