The c.e. sets disjoint from an c.e. set A

Peter Cholak and Peter Gerdes and Karen Lange

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The set up

- A and B and most mentioned (but not all) sets are always an infinite coinfinite c.e. set.
- $\mathcal{D}(A) = \{B | B \cap A =^* \varnothing\}$, the sets disjoint from A.
- $\{D_0, D_1, D_2 \ldots\}$ *generates* $\mathcal{D}(A)$ iff, for all i, $D_i \in \mathcal{D}(A)$ and, for all $B \in \mathcal{D}(A)$, there is an i such that $B \subseteq^* \bigcup_{j < i} D_i$ (B is *covered* by the union). The D_i are the *generators*.

Theorem (Main Theorem, Take 1)

The possible generators for $\mathcal{D}(A)$ break up into 10 invariant types. Moreover, all types are realized when A is a $\mathcal{D}$-maximal set.

Type 1

- A is simple iff $\mathcal{D}(A)$ is generatored by $\varnothing$. Write this as $\mathcal{D}(A) = \{\varnothing\}$.
- Simple is definable in the family of all c.e. sets with the language of inclusion, union, intersection, $\varnothing$ and ω. We call this structure $\mathcal{E}$. More on next slide.
- A and $\hat{A}$ are in the same orbit and A is simple so is $\hat{A}$.
- So invariant.

Why definable in $\mathcal{E}$?

- R is computable (always!).
- R computable iff there is a W such that $R \sqcup W = \omega$. So if R is computable then in $\mathcal{E}$, $\overline{R}$ is defined.
- Every infinite c.e. set has an infinite noncomputable subset. Fails for finite sets.
- F is finite iff, for all $W \subseteq F$, W is computable.
- S is simple iff, for all W, if $W \cap S = \varnothing$ then W is finite.
- $W_1 =^* W_2$ and $W_1 \subseteq^* W_2$ is definable in $\mathcal{E}$.

Type 2

- A is Type 2 iff R generates $\mathcal{D}(A)$ iff $\mathcal{D}(A) = \{R\}$.
- $A \sqcup R$ is a trivial split of $A \sqcup R$. One of the halfes is computable.
- $A \sqcup R$ is simple iff $\mathcal{D}(A) = \{R\}$.
- Type 2 is invariant.

Lemma
$\mathcal{G}$ are a set of generators for $A \sqcup R$ iff $\mathcal{G}$ plus R are a set of generators for A.

Trivial splits alter the generators. What about other kinds of splits?

Types 1,2,3 $\mathcal{D}$-maximal of Type 1,2,3   Types 4,5,6   Why Push?   Last 4 Types   Type 10   Anti-Friedberg splits

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Friedberg Splits

Definition

$A_0 \sqcup A_1 = A$ is a *Friedberg split* of A iff, for all W (always c.e.), if $W - A$ is not a c.e. set either are $W - A_i$.

Theorem (Friedberg)

Every noncomputable c.e. set A has a Friedberg split. Moreover a code for the split can be found effectively in the code for A.

Proof.

On board.   □

Friedberg splits and generators

Lemma
If $A_0 \sqcup A_1$ is a Friedberg split of A and a set generators of A is G then a set of generators for A_i is G plus $A_{\overline{i}}$.

Proof.
Let $D \cap A_0 =^* \varnothing$. So $D - A_0 = D$ is a c.e. set. Thus $D - A$ is a c.e. set. $D = (D - A) \sqcup (D \cap A_0) \sqcup (A \cap A_1)$. $\qquad \square$

- Only for Friedberg splits! Take a nontrivial non-Friedberg split of a simple set. So there is a W such that $W - A$ is not a c.e. set but $W - A_0$ is a c.e. set. This set is not covered by A_1.

- Converse fails. More in 2 slides.

- Friedberg splits alter the generators.

Type 3

- A is Type 3 iff there is a noncomputable W such that $\mathcal{D}(A) = \{W\}$.
- Invariant.
- Friedberg splits of simple sets are Type 3.
- What about converse? Fails!

What are there other splits?

Nontrivial non-Friedberg splits (of simple sets)? YES!! Just wait. But . . .

Question

If $f(e) = (e_0, e_1)$ is a computable function such that if W_e is not computable then $W_{e_0} \sqcup W_{e_1}$ is a nontrivial split of W_e. Then the split is always a Friedberg split of a noncomputable W_e.

As phased the answer is no. In near future we will address what the proper question should be and hopefully answer it.

Maximal sets and $\mathcal{D}$-maximal sets

Definition
M is maximal iff, for all W, either $W \subseteq^* M$ or $W \cup M =^* \omega$ (and M is not computable).

Definition
A is $\mathcal{D}$-maximal iff, for all W, there is a $D \in \mathcal{D}(A)$ such that either $W \subseteq^* A \cup D$ or $W \cup A \cup D =^* \omega$ (and A is not computable).

Theorem (Friedberg)
Maximal sets exists.

𝒟-maximal sets of Type 1

Maximal. By Sack and Martin there are complete and
incomplete maximal sets.

Theorem (Soare)

The maximal sets form an orbit.

Splits of $\mathcal{D}$-maximal sets

Lemma
If $A_0 \sqcup A_1$ is $\mathcal{D}$-maximal and A_0 is not computable then A_0 is $\mathcal{D}$-maximal.

Proof.
On board. $\qquad\square$

Lemma (Cholak, Downey, and Herrmann following Downey and Stob)
Every nontrivial split of a $\mathcal{D}$-maximal set is a Friedberg split.

Proof.
On board $\qquad\square$

𝒟-maximal set of Type 2

So noncomputable halfes of trivial splits of maximal sets are only examples of 𝒟-maximal sets of Type 2. There are incomplete and complete examples. By Soare's result they also form an orbit.

hemi-P

Definition

A_0 is *hemi-P* iff there a noncomputable A_1 such that $A_0 \sqcup A_1$ has property P.

If P is definable so is hemi-P. Note that if P is definable then so are the Friedberg splits of the sets satisfying P. In $\mathcal{E}$ to say that "$W_e - A$ is not a c.e. set" is there is not an c.e. set W such that $W \cap A = \varnothing$ and $W \cup A = W_e \cup A$.

$\mathcal{D}$-maximal set of Type 3

- So halfes of Friedberg splits of maximal sets are only examples of $\mathcal{D}$-maximal sets of Type 3. There are incomplete and complete examples.
- Since all nontrivial splits of a maximal set are Friedberg, for a $\mathcal{D}$-maximal set A, A is hemimaximal iff A is Type 3.

Theorem (Downey and Stob, Herrmann)
The hemimaximal sets form an orbit. (Needs that all nontrivial splits are Friedberg.)

Question
Is there a definable P such that the Friedberg splits are proper subclass of the nontrivial splits?

3 of 10 done

Theorem (Main Theorem, Take 2)

The possible generators for $\mathcal{D}(A)$ break up into 10 invariant types. Moreover, all types are realized when A is a $\mathcal{D}$-maximal set.

For $\mathcal{D}$-maximal sets of Type 1, 2, and 3, all equal are definable orbits and contain complete and incomplete sets.

More Types

Lemma
If $\mathcal{D}(A)$ be generatored by $W_0, W_1, \ldots W_n$ then $\mathcal{D}(A)$ is generatored by their union.

Lemma
If $D(A)$ is partially generated by infinitely many computable sets then the same sets are covered an pairwise disjoint collection of computable sets.

Proof.
On board. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Infinitely many pairwise disjoint sets

Lemma

Assume that $\mathcal{D}(A)$ is generated by infinitely many pairwise disjoints sets then we can assume that either they all are computable (Type 4) , or there is one noncomputable set (Type 5), or none of them are computable (Type 6).

Proof.

On board. □

No longer elementary definable. But in $\mathcal{L}_{\omega_1,\omega}$.

$\mathcal{D}$-maximal sets of Type 4,5, and 6

A $\mathcal{D}$-maximal A is

- Type 4 iff Herrmann
- Type 5 iff hemi-Herrmann
- Type 6 iff A has an A-special lists.

All 3 are definable orbits containing complete and
incomplete sets (Cholak, Herrmann, Downy and Cholak for
first 2, Cholak and Harrington for last).
Sets with A-special lists are still the only known example of a
definable orbit which is not an orbit under Δ_3^0
automorphisms.

4 left

Theorem (Main Theorem, Take 3)

The possible generators for $\mathcal{D}(A)$ break up into 10 invariant types. Moreover, all types are realized when A is a $\mathcal{D}$-maximal set.

For $\mathcal{D}$-maximal sets of Type 1, 2, 3, 4, 5, and 6 all equal are definable orbits and contain complete and incomplete sets.

Why push this?

Reason 1

Theorem (Herrmann and Kummer)

*There is a split of simple (actually hhsimple) set which is
𝒟-maximal. Not a trivial or Friedberg split.*

Not of first six types. What kind of split?

Reason 2

Theorem (Cholak, Downey, Harrington)

There is a c.e. set A such that the index set $\{i \in \omega \mid W_i \cong A\}$ is Σ^1_1-complete.

Theorem (Cholak and Harrington)

If A is simple then the index set $\{i \in \omega \mid W_i \cong A\}$ is arithmetical.

W_i and A are in the same orbit iff $W_i \cong A$ iff W_i and A are automorphic.

Both groups of results have something to said but $\mathcal{D}$-maximal sets. But neither group of results completely resolves how the $\mathcal{D}$-maximal sets behave. Results like the those for first 6 types would resolve the issue. For $\mathcal{D}$-maximal sets, it is enough to know where $\mathcal{D}(A)$ is sent. Hope some structural property of $\mathcal{D}(A)$ would arise and provide resolution. Still might! Just not there yet.

Reason 3

Question

Which c.e. sets are and are not automorphic to a complete set? Is it Σ_1^1 to decide if a set automorphic to a complete set?

Harrington and Soare showed there is a realizable definable property $Q(A)$ such that A is not complete. Cholak and Harrington showed whether a simple set is automorphic to a complete set is arithmetical. But still not known exactly which simple sets are automorphic to complete sets. For example, atomless r-maximal sets. But known for hhsimple sets. So perhaps like the hhsimple sets, some structural property of $\mathcal{D}(A)$, for the $\mathcal{D}$-maximals sets, would arise and provide resolution.

Huuuh . . .

Theorem (Main Theorem, Final Take)

The possible generators for $\mathcal{D}(A)$ break up into 10 invariant types. Moreover, all types are realized when A is a $\mathcal{D}$-maximal set.
For $\mathcal{D}$-maximal sets of Type 1, 2, 3, 4, 5, and 6 all equal are definable orbits and contain complete and incomplete sets.

The $\mathcal{D}$-maximal sets of Type 7, 8, 9, or 10, break up into infinitely many orbits by defining a further invariant on each type and each example of the new invariant there are complete and incomplete examples which are not know to be automorphic.

C.e. sets as generators

Lemma

Assume that $\mathcal{D}(A)$ is partially generated by an infinite list of noncomputable c.e. sets, $\{\mathcal{D}_0, D_1, \ldots\}$. Then we can assume that these sets are pairwise disjoint or nested and $D_{n+1} - D_n$ is not an c.e. set.

Last 4 types

Type 7 $\mathcal{D}(A) = \{D_0, R_0, R_1, \ldots\}$, where the R_i are infinite pairwise disjoint computable c.e. sets (this is always the case) and D is an infinite noncomputable c.e. set (and not of a lesser type).

Type 8 $\mathcal{D}(A) = \{D_0, D_1 \ldots, R_0, R_1, \ldots\}$, where the D_i are infinite pairwise disjoint noncomputable c.e. sets (and not of a lesser type).

Type 9 $\mathcal{D}(A) = \{D_0, D_1 \ldots, R_0, R_1, \ldots\}$, where the D_i are infinite nested noncomputable c.e. sets such that, for all l, $D_{l+1} - D_l$ is not an c.e. set, and and, for each i, there are infinitely many j such that $R_j - D_i$ is infinite (and not of a lesser type).

Type 10 $\mathcal{D}(A) = \{D_0, D_1 \ldots\}$, where the D_i are infinite nested noncomputable c.e. sets such that, for all l, $D_{l+1} - D_l$ is not an c.e. set (and not of a lesser type).

r-maximal sets and Type 10

Definition
A coinfinite set A is *r-maximal* iff no infinite computable splits $\overline{A}$ into two infinite pieces iff, for every computable set R, either $R \cap \overline{A} =^* \varnothing$ ($R \subseteq^* A$) or $\overline{R} \cap \overline{A} =^* \varnothing$ (so $\overline{A} \subseteq^* R$).

Supersets of r-maximal sets not almost equal to everything are r-maximal.

Lemma
Assume that A is the half of a split of r-maximal set (so A is not simple and hence cannot have Type 1) and $\mathcal{D}(A)$ is not Type 2 or 3. Then $\mathcal{D}(A)$ is Type 10.

The assumption not Type 2 or 3 is needed due trivial and Friedberg splits.

Atomless sets and $\mathcal{D}$-maximal Type 10 sets

Definition

B is *atomless* iff B does not have a maximal superset iff for every c.e. superset C, if $C \neq^* \omega$ then there is an E such that $C \subset E \subset \omega$.

Lemma

If A is $\mathcal{D}$-maximal and of Type 10 then A is half of a split of an atomless r-maximal set.

For our proof $\mathcal{D}$-maximal is needed. What kind of split is this set? Not trivial or Friedberg.

Anti-Friedberg Splits

Definition

Let $A_0 \sqcup A_1$ be a nontrivial splitting of a A. We say $A_0 \sqcup A_1$ is a *anti-Friedberg split of A* if for all W, either

1. there is a D such that $D \cap A_0 =^* \varnothing$ and $W \cup D \cup A =^* \omega$ or

2. $W - A_0$ is c.e. set.

The order of the sets, A_0 and A_1, matters. An anti-Friedberg split which is not a Friedberg split is a *properly anti-Friedberg split*.

In the first case $W - A_0$ cannot be a c.e. set or A_0 would be computable.

Anti-Friedberg Splits and $\mathcal{D}$-maximal sets

Lemma

A is $\mathcal{D}$-maximal and B is an infinite noncomputable c.e. set disjoint from A iff $A \sqcup B$ is an anti-Friedberg split.

Proof.

($\Rightarrow$) For all W there is a set D disjoint from A such that either $W \subseteq^* A \sqcup D$ (so $W - A = W \cap D$ is a c.e. set) or $W \cup A \cup D =^* \omega$.

($\Leftarrow$) For all W there is a D such that $D \cap A = \varnothing$ and either $W \cup D \cap A =^* \omega$ or $W - A = D$ is a c.e. set. In the latter case, $W \subseteq^* A \sqcup (W - A)$. $\square$

Proper anti-Friedberg splits

Lemma
Let $A \sqcup B$ be an anti-Freidberg split. (So A is $\mathcal{D}$-maximal).
$A \sqcup B$ is $\mathcal{D}$-maximal iff $A \sqcup B$ is a Friedberg split.

Proof.
($\Rightarrow$) Earlier we showed all splits of a $\mathcal{D}$-maximal are trivial or Friedberg.
($\Leftarrow$) Consider D disjoint from A. So $D - A = D$ is always a c.e. set. Since $A \sqcup B$ is a Friedberg split, $D - B = D - (A \sqcup B)$ has to also be a c.e. set. For all W, there is a D disjoint from A such that $W \subseteq^* A \sqcup B \sqcup (D - B)$ or
$W \cup (A \sqcup B \sqcup (D - B))) =^* \varnothing$. So the union is also
$\mathcal{D}$-maximal. $\square$

More examples of proper anti-Friedberg splits

If A is not simple but $\mathcal{D}$-maximal then there is an computable set R disjoint from A. Let X a noncomputable c.e. subset of R. Then R witnesses that $A \sqcup X$ is not $\mathcal{D}$-maximal. Note that $R - (A \sqcup X)$ is not a c.e. set but $R - A = R$ is a c.e. set.

Anti-Friedberg splits and generators for $\mathcal{D}(A)$

Lemma
If $A_0 \sqcup A_1 = A$ is an anti-Friedberg split of a r-maximal set A then $\mathcal{D}(A_1)$ is generatored by A_0 (so Type 3)

It is unclear what happens in the above lemma when A is hhsimple.

Two questions

Question
If $A_0 \sqcup A_1 = \widehat{A}_0 \sqcup \widehat{A}_1$ are anti-Friedberg splits then A_0 and $\widehat{A}_0$ are automorphic?

Question
If $f(e) = (e_0, e_1)$ is a computable function such that if W_e is not computable then $W_{e_0} \sqcup W_{e_1}$ is a nontrivial split of W_e. Then the split is always a Friedberg split of a noncomputable W_e.

As noted earlier, as phased the answer is no. In near future we will address what the proper question should be and hopefully answer it.