Algebraic Computability: A Personal Perspective

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- Computable Ring Theory.

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Definition

A *computable ring* is a computable subset $A \subseteq \mathbb{N}$ equipped with two computable binary operations $+$ and $\cdot$ on A, together with elements $0, 1 \in A$ such that $R = (A, 0, 1, +, \cdot)$ is a ring.

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All rings will be *countable* and *commutative*, unless we say otherwise.

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A ring R is *strongly Noetherian* if there exists a number $n \in \mathbb{N}$ such that the length of every strictly increasing chain of ideals in R is bounded by n.

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Reverse Mathematics

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$$\mathsf{ACA}_0 \longrightarrow \mathsf{WKL}_0 \longrightarrow \mathsf{RCA}_0$$

Reverse Math of Rings

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Over RCA_0, WKL_0 is equivalent to the statement "Every ring contains a prime ideal."

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Ideals in Computable Rings

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Mileti: What is the reverse mathematical strength of the theorem that says every Artinian ring is Noetherian?

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If R contains an infinite strictly increasing chain of ideals, then R also contains an infinite strictly decreasing chain of ideals.

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How much computational power does it take to go from an infinite strictly increasing chain of ideals to an infinite strictly decreasing chain of ideals?

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Corollary (RCA_0)

Theorem 1 implies WKL_0.

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The following are equivalent over RCA_0.

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5. *If R is Artinian, then $J \subset R$ exists and R/J is Noetherian.*

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Theorem (Conidis, 2010)

There is a computable ring R such that, for every $n \in \mathbb{N}$, R contains a strictly increasing chain of computable ideals of length n, and such that every infinite strictly decreasing chain of ideals in R computes $\emptyset'$.

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Theorem 2 is equivalent to ACA_0.

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Corollary (RCA_0)

Theorem 1 is implied by ACA_0*, and implies* WKL_0.

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$$Ann(x) = \{y \in R : x \cdot y = 0\} \subset R$$

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Theorem (Conidis, 2012)

Theorem 1 is equivalent to WKL_0 over RCA_0.

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A Computable Structure Theorem for Computable Artinian Rings

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Theorem (Conidis, 2013)

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$$R \supset M \supset M^2 \supset M^3 \cdots \supset M^N = 0.$$

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Corollary

WKL_0 proves Theorem 1.

Combination of two different proofs over $I\Sigma_2$, and the Computable Structure Theorem.

Computable Vector Spaces

Theorem (Downey, Hirschfeldt, Kach, Lempp, Mileti, Montalbán, 2007)

There is a computable infinite dimensional vector space V such that every nontrivial subspace of V is of PA Turing degree. The converse is obvious.

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Theorem (DHKLMM, 2007)

WKL_0 is equivalent to saying that "every vector space of dimension at least two has a nontrivial subspace."

Foundational Questions About Infinite Dimensional Vector Spaces

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Notice that (1) is weaker than (2),(3), and it is not hard to check that (1)–(3) follow from ACA_0.

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Corollary

Statements (1)–(3) from the previous slide are equivalent to ACA_0.

2-Based Vector Spaces

Definition

A (computable) vector space V is 2-based if it is a quotient of the form

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There is a computable 2-based vector space V such that every infinite/cofinite dimensional subspace of V computes $\emptyset'$. (Analogous to (3))

2-Based Vector Spaces

Corollary

The statements

- *"Every infinite dimensional 2-based vector space contains an infinite/coinfinite dimensional subspace," and*
- *"Every infinite dimensional 2-based vector space contains an infinite/cofinite dimensional subspace"*

Are equivalent to ACA_0 (over RCA_0).

Infinite Dimensional Proper Subspaces of 2-Based Vector Spaces

Theorem

Let P be a PA-Turing Degree, and let V be a computable infinite dimensional 2-based vector space. Then P computes an infinite dimensional proper subspace of V. (Analogous to (1))

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The statement "Every infinite dimensional 2-based vector space has an infinite dimensional subspace" is equivalent to WKL_0 over RCA_0.

This gives a connection between the equational definition of a vector space and its algorithmic properties. More complicated (i.e. non-2-based) equations are required to produce an infinite dimensional vector space all of whose proper infinite dimensional subspaces compute $\emptyset'$.

Euclidean Domains

Definition

A ring R is said to be a Euclidean domain if it is an integral domain and there is a function $\varphi : R \to \mathbb{N}$ such that:

1. $\varphi(x) = 0$ whenever $x = 0$ or x is a unit.
2. For all nonzero $x \in R$ and $y \in R$ we can write

$$y = qx + r$$

where $\varphi(x) > \varphi(r)$.

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Every Eulidean ring has a "Euclidean Algorithm." If R is a computable Euclidean ring with a computable Euclidean function φ, then R has a computable Euclidean Algorithm.
For the purposes of a Euclidean Algorithm, we can replace $\mathbb{N}$ with any given ordinal α. Such rings are called *Transfinite* Euclidean Domains.

Transfinite Euclidean Domains

Theorem (Motzkin, 1949)

R is a Transfinite Euclidean Domain if and only if there is an ordinal β and strictly increasing sequence of sets $\{R_\alpha\}_{\alpha<\beta}$ such that:

1. *R_0 exactly contains the zero element and units of R;*
2. *$R_{\alpha+1}$ exactly contains those elements $x \in R$ such that for all $y \in R$ there exists $r \in \cup_{\rho<\alpha+1} R_\rho$;*
3. *$R_\alpha = \cup_{\rho<\alpha} R_\rho$ whenever α is a limit ordinal;*
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Corollary (Samuel, 1971)

Every Transfinite Euclidean Domain R has a minimal Euclidean function φ_0 such that $\varphi_0(x) = \alpha$ if and only if $x \in R_\alpha$. Moreover, φ_0 is the pointwise minimum over all Euclidean functions for R.

Transfinite Euclidean Domains

Definition

The Transfinite *Euclidean Rank* of a Transfinite Euclidean Domain R is the ordinal β in the Theorem above. It is the least ordinal β for which there is a Euclidean function $\varphi : R \to \beta$.

A Question

Question (Motzkin 1949, Samuel 1971)

Is there a properly Transfinite Euclidean Domain? In other words, is there a Euclidean Domain R with Euclidean Rank $> \omega$?

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Theorem (Hiblot, 1975)

There is a Properly Transfinite Euclidean Domain of rank $\leq \omega^2$ (smallest).

A More Genreal Question

Question (Conidis, 2012)

What is the Reverse Mathematical strength of the statement (MTEF) that says "Every transfinite Euclidean ring has a minimal Transfinite Euclidean Function"?
Is there a largest possible Transfinite Euclidean Rank for all Euclidean Domains? (This generalizes Motzkin, Samuel above)

Theorem (Conidis, 2013)

Every cardinal κ is the Euclidean rank of some Transfinite Euclidean Domain R.

Thank You!