Strong reductions between combinatorial problems

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Two approaches.

There are two different but complimentary approaches for studying the computable content of mathematical principles.

Reverse mathematics: calibrates the strength of theorems according to the set-existence axioms needed to prove them.

Takes place inside subsystems of second-order arithmetic, RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0.

Effective mathematics: investigates the computational complexity of solutions to computable problems.

Locates sets within various inductive structures, e.g., the jump hierarchy, the arithmetical hierarchy, etc.

There is a fruitful interplay between these approaches.

Π^1_2 principles.

Most of the principles we study are Π^1_2 statements,

$$\forall X\, \exists Y\, \varphi(X, Y),$$

where φ is arithmetical (possibly with parameters).

Recall that for a set $X \subseteq \omega$, $[X]^n = \{F \subseteq X : |F| = n\}$.

RT^n_k. For every coloring $f : [\omega]^n \to k$, there exists an infinite $H \subseteq \omega$ such that H is homogeneous for f, i.e., f is constant on $[H]^n$.

Such principles generally have a natural class of instances, and for each instance, a natural class of solutions.

Example. For Ramsey's theorem, the instances are colorings $f : [\omega]^n \to k$, and the solutions to a given such f are the infinite homogeneous sets for f.

Π^1_2 principles.

There are many well-known examples of Π^1_2 principles.

WKL. Every infinite tree $T \subseteq 2^{<\omega}$ has an infinite path.

WWKL. Every tree $T \subseteq 2^{<\omega}$ such that for all n,

$$\frac{|\{\sigma \in T : |\sigma| = n\}|}{2^n}$$

is uniformly bounded away from 0 has an infinite path.

SRT^2_2. Every stable coloring $f : [\omega]^n \to k$ has an infinite homogeneous set.

COH. Every family of sets $\vec{S} = \langle S_n : n \in \omega \rangle$ has an infinite cohesive set, i.e., a set C such that for all n, either $C \cap S_n =^* \emptyset$ or $C \cap \overline{S_n} =^* \emptyset$.

Π^1_2 principles.

In general, an implication $\mathsf{P} \to \mathsf{Q}$ in RCA_0 can be complicated: it can appeal to P multiple times, or make non-uniform decisions.

Theorem. For every n, $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_2 \to \mathsf{RT}^n_3$.

Proof. Let $f : [\omega]^n \to 3$ be given. Define $g : [\omega]^n \to 2$ by

$$g(\mathbf{x}) = \begin{cases} 0 & \text{if } f(\mathbf{x}) = 0; \\ 1 & \text{if } f(\mathbf{x}) \in \{1, 2\}. \end{cases}$$

By RT^n_2, let H be an infinite homogeneous set for g.

If H has color 0, then H is homogeneous for f with color 0.

Otherwise, $f(\mathbf{x}) \in \{1, 2\}$ for all $\mathbf{x} \in [H]^2$. That is, $f \restriction [H]^2$ is a 2-coloring. So by RT^n_2 again, there is an infinite homogeneous set $\widetilde{H} \subseteq H$ for f.

Uniform reducibility.

In many cases, however, an implication of Π^1_2 principles is simpler, stemming from a stronger reduction holding between the two.

Definition. $\mathsf{Q} \leqslant_{\mathrm{u}} \mathsf{P}$ if there are procedures Φ and Ψ such that if A is an instance of Q then $\Phi(A)$ is an instance of P, and if S is a solution to $\Phi(A)$ then $\Psi(A \oplus S)$ is a solution to A.

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(In computable analysis, this is called (strong) Weihrauch reducibility.)

Most implications between Π^1_2 principles are strong uniform reductions that formalize. Frequently, the backwards procedure, Ψ, is the identity.

Uniform reducibility.

Examples. If $j < k$, then $\mathsf{RT}^n_j \leqslant_{\mathrm{su}} \mathsf{RT}^n_k$.

$\mathsf{SRT}^2_2 \leqslant_{\mathrm{su}} \mathsf{RT}^2_2$.

If $n < m$, then $\mathsf{RT}^n_k \leqslant_{\mathrm{su}} \mathsf{RT}^m_k$.

Cholak, Jockusch, and Slaman. $\mathsf{COH} \leqslant_{\mathrm{su}} \mathsf{RT}^2_2$.

Jockusch. If $3 \leqslant n < m$, then $\mathsf{RT}^n_k \to \mathsf{RT}^m_k$ over RCA_0 but $\mathsf{RT}^m_k \nleqslant_{\mathrm{u}} \mathsf{RT}^n_k$. Indeed, every computable instance of RT^n_k has a $\emptyset^{(n)}$-computable solution, but this is not the case for RT^m_k.

Jockusch. If $n < m$, the degrees of solutions to computable instances of DNR_n and DNR_m agree. But $\mathsf{DNR}_n \nleqslant_{\mathrm{u}} \mathsf{DNR}_m$.

Different numbers of colors.

There is no known degree-theoretic difference between solutions to computable instances of RT^n_j and RT^n_k. This motivates the following:

Question. If $j < k$, is $\mathsf{RT}^n_k \leqslant_{(\mathrm{s})\mathrm{u}} \mathsf{RT}^n_j$?

Definition. For a Π^1_2 principle P, let P^2 be the principle whose instances are pairs $\langle A, B\rangle$ with A and B instances of P, and solutions are pairs $\langle S, T\rangle$ with S a solution to A and T a solution to B.

Example. $\mathsf{COH}^2 \leqslant_{\mathrm{su}} \mathsf{COH}$.
$\mathsf{WKL}^2 \leqslant_{\mathrm{su}} \mathsf{WKL}$.
$(\mathsf{RT}^n_j)^2 \leqslant_{\mathrm{su}} \mathsf{RT}^n_{j^2}$.

It follows that if $\mathsf{RT}^n_{j^2} \leqslant_{(\mathrm{s})\mathrm{u}} \mathsf{RT}^n_j$ then $(\mathsf{RT}^n_j)^2 \leqslant_{(\mathrm{s})\mathrm{u}} \mathsf{RT}^n_j$.

Multiple applications.

Definition. For a Π^1_2 principle P, define analogously the principle P^n for each $n \geqslant 2$, and the principle P^ω.

Observation. For any P, if $\mathsf{P}^2 \leqslant_{(s)u} \mathsf{P}$ then $\mathsf{P}^n \leqslant_{(s)u} \mathsf{P}$ for all $n \geqslant \omega$.

For example, suppose $\mathsf{P}^2 \leqslant_u \mathsf{P}$ via Φ and Ψ. If $\langle A_0, A_1, A_2, A_3 \rangle$ is any instance of P^4, then

$$\Phi(A_0, \Phi(A_1, \Phi(A_2, A_3)))$$

is an instance of P. By repeatedly applying Ψ, we can unravel any solution S to this instance into a sequence of solutions to the A_i.

Multiple applications.

Question. If $\mathsf{P}^2 \leqslant_{(\mathrm{s})\mathrm{u}} \mathsf{P}$, must it be that $\mathsf{P}^\omega \leqslant_{(\mathrm{s})\mathrm{u}} \mathsf{P}$?

Given a sequence $\langle A_0, A_1, \ldots \rangle$ of colorings $[\omega]^n \to j$, we would like to imitate our argument from above by defining an instance A of P by

$$A = \Phi(A_0, \Phi(A_1, \Phi(A_2, \cdots))).$$

But this process clearly fails to converge.

However, we can solve the question by approximating this process.

For concreteness, take $\mathsf{P} = \mathsf{RT}^n_j$, and suppose $\langle \mathsf{RT}^n_j, \mathsf{RT}^n_j \rangle \leqslant_{\mathrm{u}} \mathsf{RT}^n_j$ via Φ and Ψ. Fix a sequence of colorings $f_0, f_1, \ldots : [\omega]^n \to j$.

Multiple applications.

We build new colorings $g_0, g_1, \ldots : [\omega]^n \to j$ so as to satisfy

$$g_i(\mathbf{x}) = \Phi(f_i, g_{i+1})(\mathbf{x})$$

for all large enough $\mathbf{x} \in [\omega]^n$. Thus, $g_0 \approx \Phi(f_0, g_1)$.

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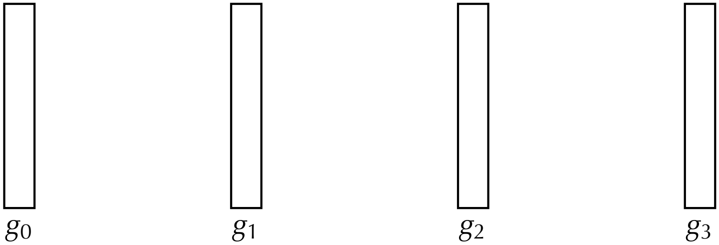

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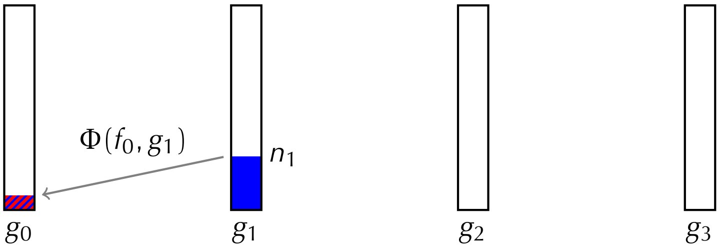

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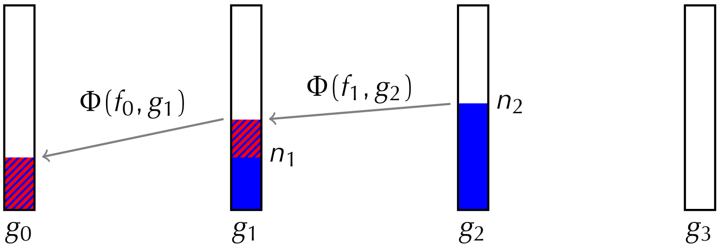

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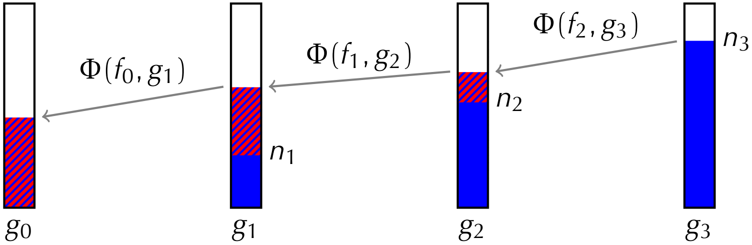

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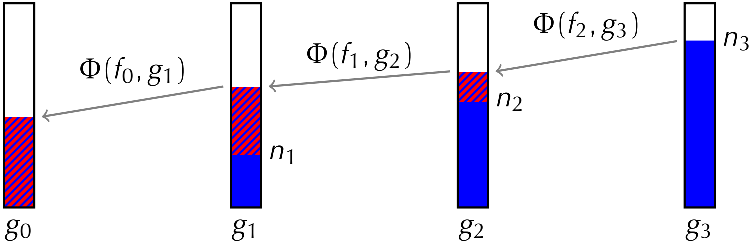

If H_i is homogeneous for g_i then $H_i - n_i$ is homogeneous for $\Phi(f_i, g_{i+1})$.

Squashing theorem.

We conclude that if $(\mathsf{RT}^n_j)^2 \leqslant_{\mathrm{u}} \mathsf{RT}^n_j$ then $(\mathsf{RT}^n_j)^\omega \leqslant_{\mathrm{u}} \mathsf{RT}^n_j$.

A similar argument holds for many other Π^1_2 principles, including SRT^2_2, COH, WKL_0, TS^2_k, etc.

Theorem (Dorais, Dzhafarov, Hirst, Mileti, and Shafer). If P is Π^1_2 and satisfies totality and finite tolerance, then $\mathsf{P}^\omega \leqslant_{\mathrm{(s)u}} \mathsf{P}$ if $\mathsf{P}^2 \leqslant_{\mathrm{(s)u}} \mathsf{P}$.

(Totality and finite tolerance are mild combinatorial assumptions.)

As given, our argument above only works for $\leqslant_{\mathrm{u}}$. This is because the numbers n_i used in the construction of the g_i depend on the instance $f_0, f_1, \ldots$ of colorings. A more careful construction is necessary for $\leqslant_{\mathrm{su}}$.

Different numbers of colors.

Theorem (Jockusch). Every computable $f\colon [\omega]^n \to j$ has an infinite homogeneous set H with $H' \leqslant_T \emptyset^{(n)}$.

But one can build a computable sequence $f_0, f_1, \ldots \colon [\omega]^n \to j$ so that every sequence of homogeneous sets computes $\emptyset^{(n)}$.

Corollary. $(\mathrm{RT}^n_j)^\omega \nleqslant_{\mathrm{u}} \mathrm{RT}^n_j$. Hence, $(\mathrm{RT}^n_j)^2 \nleqslant_{\mathrm{u}} \mathrm{RT}^n_j$, so $\mathrm{RT}^n_{j^2} \nleqslant_{\mathrm{u}} \mathrm{RT}^n_j$.

We do not know how to extend this to arbitrary $k > j$ in the case of $\leqslant_{\mathrm{u}}$.

Corollary. If $j < k$, then $\mathrm{RT}^n_k \nleqslant_{\mathrm{su}} \mathrm{RT}^n_j$.

Proof. One can show that if $\mathrm{RT}^n_k \leqslant_{\mathrm{su}} \mathrm{RT}^n_j$ then $\mathrm{RT}^n_{k^s} \leqslant_{\mathrm{su}} \mathrm{RT}^n_{j^s}$ for all s.
(This is a combinatorial argument. It is not clear if it holds also for $\leqslant_{\mathrm{u}}$.)
Take s so large that there are two different powers of 2 between j^s and k^s.

Uniform reductions and thin sets.

Definition. Fix n and $k \in \{2, 3, \dots, \omega\}$. A set $T \subseteq \omega$ is thin for a coloring $f\colon [\omega]^n \to k$ if there is a $c < k$ such that $f(\mathbf{x}) \neq c$ for all $\mathbf{x} \in [T]^n$.

TS^n_k. Every $f\colon [\omega]^n \to k$ has an infinite thin set.

For every n, we have that

$$\mathsf{TS}^n_\omega \leqslant_{\mathrm{su}} \cdots \leqslant_{\mathrm{su}} \mathsf{TS}^n_4 \leqslant_{\mathrm{su}} \mathsf{TS}^n_3 \leqslant_{\mathrm{su}} \mathsf{TS}^n_2 = \mathsf{RT}^n_2 <_{\mathrm{su}} \mathsf{RT}^n_3 <_{\mathrm{su}} \mathsf{RT}^n_4 <_{\mathrm{su}} \cdots .$$

Question (Miller). If $j < k \in \{2, 3, \dots, \omega\}$, does $\mathsf{TS}^n_k \to \mathsf{TS}^n_j$ over RCA_0?

Theorem (Dorais, Dzhafarov, Hirst, Mileti and Shafer, $n = 1$; Hirschfeldt and Joskusch, $n \geqslant 2$). If $j < k \in \{2, 3, \dots, \omega\}$, then $\mathsf{TS}^n_j \nleqslant_u \mathsf{TS}^n_k$.

The proof is a direct construction. The squashing theorem does not help.

Uniform reductions and rainbows.

Definition. Fix n, k. A coloring $f : [\omega]^n \to \omega$ is k-bounded if $|f^{-1}(c)| \leqslant k$ for all $c \in \omega$. A set $R \subseteq \omega$ is a rainbow for f if $f \restriction [R]^n$ is injective.

RRT^n_k. Every k-bounded $f : [\omega]^n \to \omega$ has an infinite rainbow.

Theorem (Csima and Mileti). If X is 2-random then every computable instance of RRT^2_k has an X-computable solution.

Theorem (Slaman). RRT^2_2 does not imply $\forall k\ \mathsf{RT}^1_k$ over RCA_0.

By contrast, for each k, RT^1_k is computably true.

Theorem (Dorais, Dzhafarov, Hirst, Mileti and Shafer). The Csima-Mileti result holds if RRT^2_k is replaced by $(\mathsf{RRT}^2_k)^\omega$. Hence, $\mathsf{RT}^1_k \not\leqslant_{\mathrm{u}} \mathsf{RRT}^2_2$.

Proof. Otherwise, $(\mathsf{RT}^1_k)^\omega \leqslant_{\mathrm{u}} (\mathsf{RRT}^2_2)^\omega$. But the former can code $\emptyset'$.

Computable (non-uniform) reducibility.

Definition. Q $\leqslant_c$ P if every instance A of Q computes an instance B of P, such that if S is a solution to B then $A \oplus S$ computes a solution to A.

Q $\leqslant_{sc}$ P if every instance A of Q computes an instance B of P, such that if S is a solution to B then S computes a solution to A.

	strong	weak
uniform	$\leqslant_{su}$	$\leqslant_u$
non-uniform	$\leqslant_{sc}$	$\leqslant_c$

$\leqslant_c$ is perhaps the most natural reduction between Π^1_2 principles.

Stable colorings and Δ^0_2 partitions.

Definition. A coloring $f : [\omega]^2 \to 2$ is stable if $\lim_y f(x, y)$ exists for all x.

SRT^2_k. Every stable $f : [\omega]^2 \to k$ has an infinite homogeneous set.

D^2_k. For every Δ^0_2-definable partition $\langle A_0, \ldots, A_{k-1} \rangle$ of ω, there is an infinite set S contained in one of the A_i.

These are equivalent under $\leqslant_{\mathrm{u}}$, essentially just by the limit lemma.

Theorem (Chong, Lempp, and Yang). Over RCA_0, $\mathsf{D}^2_k \leftrightarrow \mathsf{SRT}^2_k$.

Theorem (CJS; Mileti). Over RCA_0, $\mathsf{RT}^2_k \leftrightarrow \mathsf{D}^2_k + \mathsf{COH}$.

(That $\mathsf{RT}^2_2 \to \mathsf{COH}$ is just a formalization of the fact that $\mathsf{COH} \leqslant_{\mathrm{su}} \mathsf{RT}^2_2$.)

COH, D^2_2, and ω-models.

Theorem (Chong, Slaman, Yang). Over RCA_0, COH is not implied by D^2_2.

The proof uses a very special non-standard model of RCA_0.

Question. Is every ω-model of D^2_2 a model of COH?

The result of Chong, Slaman, and Yang suggests that if there is a proof of COH from SRT^2_2 in ω-models, it should be complicated.

For instance, their model satisfies Σ^0_2, which usually suffices to formalize finite injury arguments.

Even so, the implication itself may be the typical one.

Question. Does COH reduce to D^2_2 according to any of our notions?

Cohesive avoidance.

It would be the case that COH $\leqslant_{\mathrm{su}}$ D^2_k if, given $\vec{S} = \langle S_i : i \in \omega \rangle$, there were a partition $\langle A_0, \ldots, A_{k-1} \rangle$ of ω that is Δ^0_2 in $\vec{S}$, such that any infinite subset of any A_i computed an $\vec{S}$-cohesive set.

If $\vec{S} = \langle S_0, \ldots, S_{n-1} \rangle$, then this holds with $k = 2^n$, as any infinite subset of any of the 2^n many Boolean combinations of the S_i is $\vec{S}$-cohesive.

Theorem (Dzhafarov). Fix n and $k < 2^n$. There is a family $\vec{S} = \langle S_0, \ldots, S_{n-1} \rangle$ such that for any partition $\langle A_0, \ldots, A_{k-1} \rangle$ of ω, arithmetical in $\vec{S}$, there is an infinite subset of one of the A_i that computes no $\vec{S}$-cohesive set.

In particular, this is true for partitions that are Δ^0_2 in $\vec{S}$.

Corollary. For all k, COH $\not\leqslant_{\mathrm{sc}}$ D^2_k. In fact, COH $\not\leqslant_{\mathrm{sc}}$ $D^2_{<\infty}$.

Cohesive avoidance.

The proof of the theorem is a forcing argument, using a combinatorial elaboration of Seetapun's argument.

To make H not compute an $\vec{S}$-cohesive set, we must ensure that $\Delta(H) \cap S_i$ and $\Delta(H) \cap \overline{S_i}$ infinite, for some i that depends on Δ.

In the case of a strong reduction, we essentially build H independently of $\vec{S}$: we extend H to find a new computation $\Delta(H)(x) \downarrow = 1$, and then set $S_i(x) = 0$ or $S_i(x) = 1$ as needed, for an appropriate i.

For a weak reduction, we are looking at computations of the form $\Delta(\vec{S} \oplus H)(x) \downarrow = 1$, so the bits of S_i must be built along with H. This makes matters much more difficult.

But if we fix Δ, we can make some progress.

Cohesive avoidance.

Theorem (Dzhafarov). Fix Δ and Γ. There is a family $\vec{S} = \langle S_i : i \in \omega \rangle$ such that every stable $f \leqslant_T \vec{S}$ has either an infinite red homogeneous set R with $\Delta(\vec{S} \oplus R)$ not $\vec{S}$-cohesive, or an infinite blue homogeneous set B with $\Gamma(\vec{S} \oplus B)$ not $\vec{S}$-cohesive.

Corollary. $\mathsf{COH} \not\leq_{\mathrm{u}} \mathsf{SRT}^2_2$.

So, in which ways is COH reducible SRT^2_2?

	strong	weak
uniform	no	no
computable	no	?

La cuenta, por favor.