A computability theoretic equivalent to Vaught's conjecture.

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Note that it follows from the Continuum Hypothesis.

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We call such a real a *presentation* of $\mathcal{A}$.

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Obs: The class of presentations of models of an $L_{\omega_1,\omega}$ sentence is Borel.

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Lemma: [Scott 65] For every structure $\mathcal{A}$, there is an $L_{\omega_1,\omega}$ sentence φ
such that if $\mathcal{B} \models \varphi$, then $\mathcal{B} \cong \mathcal{A}$.

Variations of Vaught's conjecture

Perfect set variation:

Given a theory T, either T has countably many countable models,
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Thm [Becker, Kechris]: The topological Vaught's conjecture for the group S^∞ is equivalent to Vaught's conjecture for $L_{\omega_1,\omega}$.

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- So $|\{\text{models of } T\}| \leq \aleph_1$.

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Note: This definition is independent of whether CH holds or not.

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Projective Determinacy (PD): Every projective set is determined.

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  ($0^{(\alpha)}$ is the αth Turing jump of 0.)
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- $X = \{n \in \omega : \varphi(n)\}$, where φ is a computable infinitary formula.

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  ($0^{(\alpha)}$ is the αth Turing jump of 0.)
- $X \in L(\omega_1^{CK})$.
- $X = \{n \in \omega : \varphi(n)\}$, where φ is a computable infinitary formula.

A set satisfying the conditions above is said to be hyperarithmetic.

Obs: For instance, all arithmetic sets are hyperarithmetic.

Another word on coding

Example: A countable ordered group $\mathcal{A} = (A, \times_A, \leq_A)$
can be encoded by three sets: $A \subseteq \mathbb{N}$, $\times_A \subseteq \mathbb{N}^3$ and $\leq_A \subseteq \mathbb{N}^2$.

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Let $\mathbb{K}$ be a class of structures.

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Theorem ([M.] (ZFC+PD))

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The following are equivalent:

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- If s is a strategy for I, $\{s(Y_0 \oplus s) : Y_0 \in 2^\omega\}$ is a perfect pointed tree $\subseteq P$.

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The Main Theorem–once more

Theorem ([M.] (ZFC+PD))

Let T be a theory with uncountably many countable models. The following are equivalent:

- *T is a counterexample to Vaught's conjecture.* $\Rightarrow$
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Obs: For every structure $\mathcal{A}$, $\rho(\mathcal{A}) \leq \omega_1^{\mathcal{A}} + 1$.

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Let $(\mathfrak{K}, \equiv, r)$ be scattered projective ranked equivalence relation such that $\forall Z \in \mathfrak{K},\ r(Z) < \omega_1^Z$.
For every X on a cone, (i.e. $\exists Y \forall X \geq_T Y$,) every equivalence class with an X-hyperarithmetic member has an X-computable member.

Corollary: (ZFC+PD)
If T is scattered, the class of models of T of *low Scott rank* satisfies hyperarithmetic-is-recursive on a cone.

where:

Def: $\rho(\mathcal{A})$ is the least α such that if $\mathcal{B} \equiv_\alpha \mathcal{A}$, then $\mathcal{B} \cong \mathcal{A}$.
$\omega_1^{\mathcal{A}} = \text{least}\{\omega_1^X : X \text{ computes a copy of } \mathcal{A}\}$.

Obs: For every structure $\mathcal{A}$, $\rho(\mathcal{A}) \leq \omega_1^{\mathcal{A}} + 1$.

Def: $\mathcal{A}$ has *low Scott rank* if $\rho(\mathcal{A}) < \omega_1^{\mathcal{A}}$.

A proof of hyp-is-rec

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- For that Y, $f(Y) \neq \alpha$.

The Main Theorem– yet again

Theorem ([M.] (ZFC+PD))

Let T be a theory with uncountably many countable models.
The following are equivalent:

- *T is a counterexample to Vaught's conjecture.* $\Leftarrow$
- *T satisfies hyperarithmetic-is-recursive on a cone.* $\Rightarrow$
- *There exists an oracle relative to which*

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Theorem

If T has strictly more than $\aleph_1$ many models,
then, relative to every X on a cone,
T has an X-hyperarithemetic model without an X-computable copy.

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If $\varphi_i(\bar{c}) \equiv \bigvee_j \exists(\bar{x})\psi_{i,j}(\bar{c},\bar{x})$, search for j, $\mathcal{A}_s \in \mathbb{K}^{fin}$ and $\bar{a} \in \mathcal{A}_s$
such that $\mathcal{A}_{s-1} \subseteq \mathcal{A}_s$ and $\mathcal{A}_s \models \psi_{i,j}(\bar{c},\bar{a})$.

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- Assuming T is Π_α^c, if $\mathcal{A} \models T$, then $\mathcal{A}^{(\alpha)}$ computes $\hat{\mathcal{A}}$ with $\hat{\mathcal{A}} \models \hat{T}$.

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$\equiv_\alpha$ is Borel, so, by [Silver 80], $\mathbf{bf}_\alpha(\mathbb{K})$ has size either countable or continuum.

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Complete sets of Π^{in}_{α}-formulas

Definition ([M])

$\{P_0, P_1, ...\}$ is a *complete set of Π^{in}_{α} formulas for* $\mathbb{K}$ if every $\Sigma^{in}_{\alpha+1}$ $\mathcal{L}$-formula is equivalent to a Σ^{in}_{1} $(\mathcal{L} \cup \{P_0, ...\})$-formula.

Example: Π_1^c relations on Linear orderings

Let $\mathbb{K}$ = linear orderings.
Let $Succ = \{(a, b) \in \mathcal{A}^2 : a < b \,\&\, \nexists c \ (a < c < b)\}$.

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Every Σ^c_2 formula is equivalent in $\mathbb{K}$ to a
$0'$-disjunction of Σ_1 finitary formulas in the language $\{\leq, Succ\}$.

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Every Σ_3^c formula is equivalent to a $0^{(2)}$-disjunction of
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Boolean algebras

Theorem ([Harris, M] rels. used by Downey-Jockusch, Thurber, Knight-Stob)

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Furthermore, $\forall n$ there is a finite complete set of Π^c_n relations

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Def: A set D is *weakly* Σ^0_α *coded in* $\mathcal{A}$ if D is left-Σ^0_α in every copy of $\mathcal{A}$.

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Let $\mathbb{K}$ be a Borel class of countable structures.
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The Main Theorem– so you don't forget

Theorem ([M.] (ZFC+PD))

Let T be a theory with uncountably many countable models.
The following are equivalent:

- *T is a counterexample to Vaught's conjecture.* $\Leftarrow$
- *T satisfies hyperarithmetic-is-recursive on a cone.* $\Rightarrow$
- *There exists an oracle relative to which*

$$\{Sp(\mathcal{A}) : \mathcal{A} \models T\} = \{\{X \in 2^\omega : \omega_1^X \geq \alpha\} : \alpha \in \omega_1\}.$$

Theorem

If $|\mathbf{bf}_\alpha(\mathbb{K})| = 2^{\aleph_0}$ for some α, then, relative to every X on a cone, $\mathbb{K}$ has an X-hyperarithemetic model without an X-computable copy.

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Recall: For $\beta < \alpha$ and $p \in \mathbf{bf}_\beta(\mathbb{K})$, $\mathcal{A} \models R_p(\bar{x}) \iff \mathcal{A} \models \varphi(\bar{x})$, for every $\varphi(\bar{x}) \in p$.

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So, $\mathcal{A}$ does not have an X-computable copy.

The Main Theorem–again

Theorem ([M.] (ZFC+PD))

Let T be a theory with uncountably many countable models. The following are equivalent:

- *T is a counterexample to Vaught's conjecture.*
- *T satisfies hyperarithmetic-is-recursive on a cone.*
- *There exists an oracle relative to which*

$$\{Sp(\mathcal{A}) : \mathcal{A} \models T\} \; = \; \{\{X \in 2^\omega : \omega_1^X \geq \alpha\} : \alpha \in \omega_1\}.$$

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Def: $Sp(\mathcal{A}) = \{X \in 2^\omega : X$ computes a copy of $\mathcal{A}\}$.

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Theorem ([M.] (ZFC+PD))

If T is a counterexample to Vaught's conjecture, then, there is $Y \in 2^\omega$ such that, for every $C \subseteq 2^{\mathbb{N}}$, the following are equivalent:

- $C = Sp^Y(\mathcal{A})$ *for some* $\mathcal{A} \models T$,
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Corollary: If T is a counterexample to Vaught's conjecture,
then T satisfies hyperarithmetic-is-recursive on a cone.

An observation

Suppose you have a proof of:

"If you have a structure $\mathcal{A}$ with $Sp(\mathcal{A}) = Sp(\omega_1^{CK})$, there exists continuum many $\mathcal{B}$ with $\mathcal{B} \models \Pi_2^c\text{-}Th(\mathcal{A})$."

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Question: What can we say about the structures with $Sp(\mathcal{A}) = Sp(\omega_1^{CK})$?

Structures with the same spectrum as ω_1^{CK}

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In all these examples, we know that
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Thm: [Harrison '66] There is a computable linear ordering $\mathcal{H}$ such that
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i.e. $\mathcal{H}^X \cong \omega_1^X + \omega_1^X \cdot \mathbb{Q}$ has no X-hyp descending sequences.

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Let $\mathbb{K} = \{\mathbb{Z}^\alpha \cdot \mathbb{Q} : \alpha < \omega_1\}$ as linear orders.

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($\subseteq$): if X computes $\mathbb{Z}^\alpha \cdot \mathbb{Q}$, it computes $\mathbb{Z}^\beta$ for every $\beta < \alpha$. But X computes no copy of $\mathbb{Z}^{\omega_1^X}$, because the tree $\{p \colon (2^{<n}, \leq_{lex}) \to \mathbb{Z}^\beta\}$ is WF and has rank $\geq \beta$.)

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Gandy's Basis Theorem:
If φ is Σ_1^1, and $\exists X \varphi(X)$, then there is such an X with $\omega_1^X = \omega_1^{CK}$.

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Lemma: If $\omega_1^X = \omega_1^Y$, then there is G such that

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From now on, we work relative to the base of this cone.

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- Both $\mathcal{B}$ and $\mathcal{A}$ are computable in $X \oplus Y$, and $\mathcal{A} \equiv_{\omega_1^{X\oplus Y}} \mathcal{B}$. So $\mathcal{A} \cong \mathcal{B}$.

$\mathcal{A}$ has an X-computable model, for some X with $\omega_1^X = \omega_1^Y \leq \omega_1^{X\oplus Y}$.
Let G be such that $\omega_1^X = \omega_1^{X\oplus G} = \omega_1^G = \oplus\omega_1^{G\oplus Y} = \omega_1^Y$.

Almost there

Lemma: Suppose $\omega_1^{\mathcal{A}} \leq \omega_1^Y$ and $\mathcal{A} \models T$. Then Y computes a copy of $\mathcal{A}$, provided $\mathcal{A}$ has an X-computable model, with $\omega_1^X = \omega_1^Y = \omega_1^{X \oplus Y}$.

- Y computes an α^*-bf-structure $\mathbb{B}$, correct up to ω_1^Y, with $\alpha^* \in \mathcal{H}^Y \smallsetminus \omega_1^Y$.
- $\mathcal{A}$ satisfies $\mathbb{B}$ up to some $\beta^* \in \alpha^* \smallsetminus \omega_1^Y$, because ω_1^Y is not $\Sigma_1^1(X \oplus Y)$.
- Let p^* be such that $\mathcal{A} \models R_{p^*}^{=}$, and $\beta_{p^*} = \beta^*$.
- Y computes $\mathcal{B}$ which satisfies $\mathbb{B} \upharpoonright \beta^*$, and has same β^*-type as $\mathcal{A}$.
- So, $\mathcal{A} \equiv_{\omega_1^Y} \mathcal{B}$.
- Both $\mathcal{B}$ and $\mathcal{A}$ are computable in $X \oplus Y$, and $\mathcal{A} \equiv_{\omega_1^{X \oplus Y}} \mathcal{B}$. So $\mathcal{A} \cong \mathcal{B}$.

$\mathcal{A}$ has an X-computable model, for some X with $\omega_1^X = \omega_1^Y \leq \omega_1^{X \oplus Y}$.
Let G be such that $\omega_1^X = \omega_1^{X \oplus G} = \omega_1^G = \oplus\omega_1^{G \oplus Y} = \omega_1^Y$.
Apply the lemma above twice.

The two steps

Suppose T is a scattered theory with uncountably many models.
We want to show:

There is an oracle relative to which

1. For every admissible α, there is $\mathcal{A} \models T$ with $\omega_1^{\mathcal{A}} = \alpha$.
2. For every $\mathcal{A} \models T$, $Sp(\mathcal{A}) = \{X : \omega_1^X \geq \omega_1^{\mathcal{A}}\}$.

As we would then get:

$$\{Sp(\mathcal{A}) : \mathcal{A} \models T\} \;=\; \{\{X \in 2^\omega : \omega_1^X \geq \alpha\} : \alpha \in \omega_1\}.$$

The main theorem–for the last time

Theorem ([M.] (ZFC+PD))

Let T be a theory with uncountably many countable models.
The following are equivalent:

- *T is a counterexample to Vaught's conjecture.*
- *T satisfies hyperarithmetic-is-recursive on a cone.*
- *There exists an oracle relative to which*

$$\{Sp(\mathcal{A}) : \mathcal{A} \models T\} \;=\; \{\{X \in 2^\omega : \omega_1^X \geq \alpha\} : \alpha \in \omega_1\}.$$