Differentiability and porosity

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"Almost everywhere" theorems

Several important theorems in analysis assert a property for almost every real z. Two examples:

Theorem (Lebesgue, 1904)

Let $E \subseteq [0,1]$ *be measurable. Then for almost every* $z \in [0,1]$*:*
if $z \in E$*, then* E *has density* 1 *at* z*.*

Theorem (Lebesgue, 1904)

Let $f : [0,1] \to \mathbb{R}$ *be of bounded variation.*
Then the derivative $f'(z)$ *exists for almost every real* z*.*

Variation of a function

Recall that for a function $g\colon [0,1] \to \mathbb{R}$ we let

$$V(g,[0,x]) = \sup \sum_{i=1}^{n-1} |g(t_{i+1}) - g(t_i)|,$$

where the sup is taken over all $t_1 \leq t_2 \leq \ldots \leq t_n$ in $[0,x]$.

We say that g is of bounded variation if $V(g,[0,1])$ is finite.

Complexity of the exception set

Theorem (Demuth 1975/Brattka, Miller, Nies 2011)

Let $r \in [0,1]$. Then
r is ML-random $\iff$
$f'(r)$ exists, for each function f of bounded variation such that $f(q)$ is a computable real, uniformly in each rational q.

- The implication "$\Rightarrow$" is an effective version of the classical theorem.
- The implication "$\Leftarrow$" has no classical counterpart. To prove it, one builds a computable function f of bounded variation that is only differentiable at ML-random reals.

Computable randomness

Can you bet on this and make unbounded profit?

10100111000101111010101000010101101111011000010111101010
10010101100011111010110001100111111101100000111001111000
00110011011110100011110100011100101011011001011100010110
01100110001111000010011001011101100100101000001110001111
11100100011000101111110100010111110011011100100110011010
00111111011010101101001101010110000011000001001101011100
01001001001011010001010000110100010100011100001100000100
11000111110111001000011001011010100111101111010101111111
00000001010011110010000000011011001010011010101101000010 ...

We call a sequence of bits computably random if no computable betting strategy (martingale) has unbounded capital along the sequence.
ML-random ⇒ computably random, but not conversely.

Computable randomness and differentiability

Theorem (Brattka, Miller, Nies, 2011)

Let $r \in [0,1]$. Then
r (in binary) is computably random $\iff$
$f'(r)$ exists, for each *nondecreasing* function f that is uniformly computable on the rationals.

- Full computability of a function $f\colon [0,1] \to \mathbb{R}$ means that with a Cauchy name for x as an oracle, one can compute a Cauchy name for $f(x)$.
- For *continuous* nondecreasing functions, full computability is equivalent to being computable on the rationals.

Other notions of effectiveness

Variants of the Demuth/ BMN theorems have been proved:

Theorem (Freer, Kjos, Nies, Stephan, 2012)

x is computably random $\Leftrightarrow$
each computable Lipschitz functions is differentiable at x.

Theorem (BMN, 2011)

z is weakly 2-random $\Leftrightarrow$
each a.e. differentiable computable function f is differentiable at z.

Theorem (Pathak, Rojas, Simpson 2011/ Freer, Kjos, Nies, Stephan, 2012)

z is Schnorr random $\Leftrightarrow$
z is a weak Lebesgue point of each L_1-computable function.

We will look at nondecreasing functions, but
vary the notion of effectiveness.

Hyperarithmetical functions

Effectiveness much higher up...

We say that z is Δ^1_1 random if no hyperarithmetical martingale succeeds on z.

Theorem

z is Δ^1_1 random

$\Leftrightarrow$ each nondecreasing hyperarithmetical f is differentiable at z

$\Leftrightarrow$ each hyperarithmetical f of bounded variation is differentiable at z.

This is because $V(f, [0, x])$ can be evaluated by quantifying over rationals, and hence is also hyperarithmetical. So the Jordan decomposition of a hyp f is hyp.

It is hard to get past Δ^1_1 randomness. Even the a.e. differentiable hyp functions only need that.

The two theorems

Firstly, we will look at feasibly computable nondecreasing functions. We obtain an analog of the Brattka, Miller, N 2011 result.

Theorem

$r \in [0,1]$ is polynomial time random $\iff$ $g'(r)$ exists, for each nondecreasing function g that is polynomial time computable.

Secondly, we look at a class of nondecreasing functions larger than computable. We say a nondecreasing function f is interval c.e. if $f(0) = 0$, and for any rational $q > p$, $f(q) - f(p)$ is a uniformly left-c.e. real.

Theorem

Let $z \in [0,1]$. Then z is a ML-random density-one point $\iff$ $f'(r)$ exists, for each interval-c.e. function f

Density, porosity, and derivatives

Density ...

The (lower Lebesgue) density of a set $\mathcal{C} \subseteq \mathbb{R}$ at a point z is the quantity

$$\varrho(\mathcal{C}|z) := \liminf_{z \in I \,\wedge\, |I| \to 0} \frac{\lambda(I \cap \mathcal{C})}{|I|},$$

where I ranges over intervals containing z.

Definition (Bienvenu, Hölzl, Miller, N, 2011)

We say that $z \in [0,1]$ is a density-one point if $\varrho(\mathcal{C}|z) = 1$ for every effectively closed class $\mathcal{C}$ containing z.

… and porosity

We say that a set $\mathcal{C} \subseteq \mathbb{R}$ is *porous at z* via the porosity factor $\varepsilon > 0$ if there exists arbitrarily small $\beta > 0$ such that $(z - \beta, z + \beta)$ contains an open interval of length $\varepsilon\beta$ that is disjoint from $\mathcal{C}$.

Definition

We call z a porosity point if some effectively closed class to which it belongs is porous at z. Otherwise, z is a non-porosity point.

Theorem (Bienvenu, Hölzl, Miller, N, 2011)

Any ML-random porosity point is Turing complete.

Dyadic versus full

A (closed) basic dyadic interval has the form $[r2^{-n}, (r+1)2^{-n}]$ where $r \in \mathbb{Z}, n \in \mathbb{N}$. For the lower dyadic density of a set $\mathcal{C} \subseteq \mathbb{R}$ at a point z only consider basic dyadic intervals containing z:

$$\varrho_2(\mathcal{C}|z) := \liminf_{z \in I \,\wedge\, |I| \to 0} \frac{\lambda(I \cap \mathcal{C})}{|I|},$$

where I ranges over basic dyadic intervals containing z.

Theorem (Khan and Miller, 2012)

Let z be a ML-random dyadic density-one point. Then z is a full density-one point.

We know from Franklin.Ng 2010 and BHMN 2011 that z is a non-porosity point. The actual statement Joe and Mushfeq proved:

Suppose z is a non-porosity point. Let $\mathcal{P}$ be a Π^0_1 class, $z \in \mathcal{P}$, and $\varrho_2(\mathcal{P} \mid z) = 1$. Then already $\varrho(\mathcal{P} \mid z) = 1$. (Same $\mathcal{P}$.)

Suppose z is a non-porosity point. Let $\mathcal{P}$ be a Π^0_1 class, $z \in \mathcal{P}$, and $\varrho_2(\mathcal{P} \mid z) = 1$. Then already $\varrho(\mathcal{P} \mid z) = 1$.

Proof.

Consider an arbitrary interval I with $z \in I$ and $\lambda_I(\mathcal{P}) < 1 - \epsilon$. Let $\delta = \epsilon/4$.

Let n be such that $2^{-n+1} > |I| \geq 2^{-n}$. Cover I with three consecutive basic dyadic intervals A, B, C of length 2^{-n}.
Say $z \in B$. Since $\mathcal{P}$ is relatively sparse in I, but thick in B, this means it must be sparse in A or C.
Let the Π^0_1 class $\mathcal{Q}$ consist of the basic dyadic intervals where $\mathcal{P}$ is thick:

$$\mathcal{Q} = [0,1] - \bigcup\{L\colon\ \lambda_L(\mathcal{P}) < 1 - \delta\}$$

where L ranges over *open* basic dyadic intervals. Then $\mathcal{Q}$ is porous at z with porosity factor $1/3$: if $z \in B$, say, then one of A, C must be missing. □

Upper and lower derivatives

Let $f\colon [0,1] \to \mathbb{R}$. We define

$$\begin{aligned} \overline{D}f(z) &= \limsup_{h\to 0} \frac{f(z+h)-f(z)}{h} \\ \underline{D}f(z) &= \liminf_{h\to 0} \frac{f(z+h)-f(z)}{h} \end{aligned}$$

Then

$$f'(z) \text{ exists} \iff \overline{D}f(z) \text{ equals } \underline{D}f(z) \text{ and is finite.}$$

Notation for slopes, and for basic dyadic intervals

For a function $f\colon \subseteq \mathbb{R} \to \mathbb{R}$, the *slope* at a pair a, b of distinct reals in its domain is

$$S_f(a,b) = \frac{f(a) - f(b)}{a - b}.$$

For an interval A with endpoints a, b, we also write $S_f(A)$ instead of $S_f(a,b)$.

- Let $[\sigma]$ denote the closed basic dyadic interval $[0.\sigma, 0.\sigma + 2^{-|\sigma|}]$, for a string σ.
- The open basic dyadic interval is denoted (σ).
- We write $S_f([\sigma])$ with the expected meaning.

Pseudo-derivatives

- If f is only defined on the rationals in $[0,1]$, we can still consider the upper and lower *pseudo*-derivatives defined by:

$$\underset{\sim}{D} f(x) = \liminf_{h \to 0^+} \{S_f(a,b) \mid a \leq x \leq b \wedge 0 < b - a \leq h\},$$
$$\widetilde{D} f(x) = \limsup_{h \to 0^+} \{S_f(a,b) \mid a \leq x \leq b \wedge 0 < b - a \leq h\}.$$

  where a, b range over rationals in $[0,1]$.
- If f is total and continuous, or nondecreasing, this is the same as the usual derivatives.
- We will use the subscript 2 to indicate that all the limit operations are restricted to the case of basic dyadic intervals containing z. For instance,

$$\widetilde{D}_2 f(x) = \limsup_{|\sigma| \to \infty} \{S_f([\sigma]) \mid x \in [\sigma]\}.$$

Slopes and martingales

The basic connections:

- if f is nondecreasing then $M(\sigma) = S_f([\sigma])$ is a martingale.
- M succeeds on $z \Leftrightarrow \widetilde{D}_2 f(z) = \infty$
- M converges on $z \Leftrightarrow \underset{\sim}{D}_2 f(z) = \widetilde{D}_2 f(z) < \infty$

High dyadic slopes lemma

Suppose $f\colon [0,1] \to \mathbb{R}$ is a nondecreasing function. Suppose for a real $z \in [0,1]$ we have

$$\widetilde{D}_2 f(z) < p < \widetilde{D} f(z).$$

Let $\sigma^* \prec Z$ be any string such that $\forall \sigma\, [Z \succ \sigma \succeq \sigma^* \Rightarrow S_f([\sigma]) \leq p]$. Then the closed set

$$\mathcal{C} = [\sigma^*] - \bigcup \{(\sigma)\colon\ \sigma \succeq \sigma^* \ \wedge\ S_f([\sigma]) > p\},$$

which contains z, is porous at z.

Low dyadic slopes lemma

Suppose $f\colon [0,1] \to \mathbb{R}$ is a nondecreasing function and $z \in [0,1]$ such that $\underset{\sim}{D} f(z) < q < \underset{\sim}{D}_2 f(z)$. Let $\sigma^* \prec Z$ be any string such that $\forall \sigma\, [Z \succ \sigma \succeq \sigma^* \Rightarrow S_f([\sigma]) \geq q]$. Then the closed set

$$\mathcal{C} = [\sigma^*] - \bigcup \{(\sigma)\colon\ \sigma \succeq \sigma^* \ \wedge\ S_f([\sigma]) < q\},$$

which contains z, is porous at z.

Proof of high dyadic slopes lemma

- Suppose $k \in \mathbb{N}$ is such that $p(1 + 2^{-k+1}) < \widetilde{D}f(z)$.
- We show that there exists arbitrarily large n such that some basic dyadic interval of length 2^{-n-k} has slope $> p$, and is contained in $[z - 2^{-n+2}, z + 2^{-n+2}]$.
- In particular, we can choose 2^{-k-2} as a porosity constant.

Proof.

- There is an interval $I \ni z$ of arbitrarily short positive length such that $p(1 + 2^{-k+1}) < S_f(I)$. Let n be such that $2^{-n+1} > |I| \geq 2^{-n}$.
- Let a_0 be greatest of the form $v2^{-n-k}$, $v \in \mathbb{Z}$, such that $a_0 < \min I$.
- Let $a_v = a_0 + v2^{-n-k}$. Let r be least such that $a_r \geq \max I$.

By the averaging property of slopes and since f is nondecreasing, there must be an $[a_i, a_{i+1}]$ with a slope $> p$. This interval does not contain z. □

Polynomial time randomness and differentiability

Special Cauchy names

A Cauchy name is a sequence of rationals $(p_i)_{i \in \mathbb{N}}$ such that $\forall k > i \, |p_i - p_k| \leq 2^{-i}$. We represent a real x by a Cauchy name converging to x.

For feasible analysis, we use a compact set of Cauchy names: the signed digit representation of a real. Such Cauchy names, called special, have the form

$$p_i = \sum_{k=0}^{i} b_k 2^{-k},$$

where $b_k \in \{-1, 0, 1\}$. (Also, $b_0 = 0, b_1 = 1$.)

So they are given by paths through $\{-1, 0, 1\}^\omega$, something a resource bounded TM can process. We call the b_k the symbols of the special Cauchy name.

Polynomial time computable functions

The following has been formulated in equivalent forms by Ker-i-Ko (1989), Weihrauch (2000), Braverman (2008), and others.

Definition

A function $g\colon [0,1] \to \mathbb{R}$ is polynomial time computable if there is a polynomial time TM turning every special Cauchy name for $x \in [0,1]$ into a special Cauchy name for $g(x)$.

This means that the first n symbols of $g(x)$ can be computed in time poly(n), thereby using polynomially many symbols of the oracle tape holding x.

Functions such as $e^x, \sin x$ are polynomial time computable.

Analysis gives us rapidly converging approximation sequences, such as $e^x = \sum_n x^n/n!$. As Braverman points out, e^x is computable in time $O(n^3)$. Namely, from $O(n^3)$ symbols of x we can in time $O(n^3)$ compute an approximation of e^x with error $\leq 2^{-n}$.

Polynomial time randomness

A martingale $M\colon 2^{<\omega} \to \mathbb{R}$ is called polynomial time computable if from string σ and $i \in \mathbb{N}$ we can in time polynomial in $|\sigma| + i$ compute the i-th component of a special Cauchy name for $M(\sigma)$.

In this case we can compute a polynomial time martingale in base 2 dominating M (Schnorr / Figueira-N).

We say Z is polynomial time random if no polynomial time martingale succeeds on Z.

Fact

f is a nondecreasing polynomial time computable function

$$\Rightarrow$$

the slope $S_f([\sigma])$ determines a polynomial time computable martingale.

This is so because we can compute f with sufficiently high precision.

The first theorem

Theorem

The following are equivalent.

(I) $z \in [0,1]$ is polynomial time random

(II) $f'(z)$ exists, for each nondecreasing function f that is polynomial time computable.

(II) $\to$ (I)

One actually shows: z not polytime random $\Rightarrow$

$$\underline{D}f(z) = \infty \text{ for some polynomial time computable function } f.$$

This is uses machinery from the Figueira/N (2013) paper 'Randomness, feasible analysis, and base invariance'.

Proof.

- If z is not polytime random, some polytime martingale with the savings property succeeds on z.
- Then $\mathsf{cdf}_M\colon [0,1] \to \mathbb{R}$ is polytime computable (using the almost Lipschitz property).
- And the lower derivative $\underline{D}\mathsf{cdf}_M(z) = \infty$.

□

(I) → (II)

We need to show:

$$z \in [0,1] \text{ is polynomial time random} \Rightarrow f'(z) \text{ exists},$$

for each nondecreasing function f that is polynomial time computable.

- Consider the polynomial time computable martingale
$$M(\sigma) = S_f(0.\sigma, 0.\sigma + 2^{-|\sigma|}) = S_f([\sigma]) \ .$$
- $\lim_n M(Z\restriction_n)$ exists and is finite for each polynomially random Z. This is a version of Doob martingale convergence.
- Returning to the language of slopes, the convergence of M on Z means that $\underset{\sim}{D}_2 f(z) = \widetilde{D}_2 f(z) < \infty$.

Assume for a contradiction that $f'(z)$ fails to exist. First suppose that

$$\widetilde{D}_2 f(z) < p < \widetilde{D} f(z).$$

We may suppose $S_f(A) < p$ for all dyadic intervals containing z. Choose k with $p(1 + 2^{-k+1}) < \widetilde{D} f(z)$.

By the "high dyadic slopes" lemma and its proof, there exists arbitrarily large n such that some basic dyadic interval $[\tau_n]$ of length 2^{-n-k} has slope $> p$ and is contained in $[z - 2^{-n+2}, z + 2^{-n+2}]$. Let $0.Z = z$ where $Z \in 2^{\mathbb{N}}$.

Lucky case: there are infinitely many n with $\eta = Z\restriction_{n-4} \prec \tau_n$. Then the martingale that from η on bets everything on the strings of length $n + k$ other than τ_n gains a fixed factor $2^{k+4}/(2^{k+4} - 1)$.

Unlucky case: for almost all n we have $Z\restriction_{n-4} \not\prec \tau_n$. That means $0.\tau_n$ is on the left side of z, and the martingale can't use it as it may be far from Z in Cantor space!

Morayne-Solecki trick

The following was used in a paper by Morayne and Solecki (1989). They gave a martingale proof of Lebesgue differentiation theorem. For $m \in \mathbb{N}$ let $\mathcal{D}_m$ be the collection of intervals of the form

$$[k2^{-m}, (k+1)2^{-m}]$$

where $k \in \mathbb{Z}$. Let $\mathcal{D}'_m$ be the set of intervals $(1/3) + I$ where $I \in \mathcal{D}_m$.

Fact

Let $m \geq 1$. If $I \in \mathcal{D}_m$ and $J \in \mathcal{D}'_m$, then the distance between an endpoint of I and an endpoint of J is at least $1/(3 \cdot 2^m)$.

To see this: assume that $k2^{-m} - (p2^{-m} + 1/3) < 1/(3 \cdot 2^m)$. This yields $(3k - 3p - 2^m)/(3 \cdot 2^m) < 1/(3 \cdot 2^m)$, and hence $3|2^m$, a contradiction.

Using this trick

So, in the unlucky case, we instead bet on the dyadic expansion Y of $z - 1/3$. (We may assume that $z > 1/2$).

Given $\eta' = Y \upharpoonright_{n-4}$, where n is as above, we look for an extension $\tau' \succ \eta'$ of length $n + k + 1$, such that $1/3 + [\tau'] \subseteq [\tau]$ for a string $[\tau]$ with $S_f([\tau]) > p$. If it is found, we bet everything on the other extensions of η' of that length. We gain a fixed factor $2^{k+5}/(2^{k+5} - 1)$.

So we get a polytime martingale that wins on $z - 1/3$. Since polytime randomness is base invariant, this gives a contradiction.

The case $\underset{\sim}{D}f(z) < \underset{\sim}{D}_2 f(z)$ is analogous, using the low dyadic slopes lemma instead.

Ambos-Spies et al., 1996 called a martingale “weakly simple” if it has only have finitely many, rational, betting factors. The martingales showing that dyadic derivative = full derivative are such. So being polynomially stochastic is sufficient for this.

Martin-Löf random density-one points

and differentiability

The second theorem

Theorem

Let $f\colon [0,1] \to \mathbb{R}$ be an interval-c.e. function. Let z a be ML-random density-one point. Then $f'(z)$ exists.

Interval-c.e. functions

Definition

A non-decreasing function f on $[0,1]$ with $f(0)=0$ is called interval-c.e. if $f(q)-f(p)$ is a left-c.e. real uniformly in rationals $p<q$.

If f is continuous, this implies lower semicomputable.
Recall that for $g\colon [0,1]\to\mathbb{R}$ we let

$$V(g,[0,x]) = \sup \sum_{i=1}^{n-1} |g(t_{i+1})-g(t_i)|,$$

where the sup is taken over all $t_1 \le t_2 \le \ldots \le t_n$ in $[0,x]$.

Theorem (Freer, Kjos-Hanssen, N, Stephan, Rute 2012)

A continuous function f is interval-c.e. $\Leftrightarrow$
there is a computable function g such that $f(x)=\mathrm{Var}(g,[0,x])$.

Left-c.e. martingales

Definition

A martingale $M: 2^{<\omega} \to \mathbb{R}$ is called left-c.e. if $M(\sigma)$ is a left-c.e. real uniformly in string σ.

Z is ML-random iff no left-c.e. martingale succeeds on Z.

Definition

A martingale M converges on $Z \in 2^{\mathbb{N}}$ if $lim_n M(Z\restriction_n)$ exists and is finite.

$Z \in 2^{\mathbb{N}}$ is a convergence point for left-c.e. martingales if each left-c.e. martingale converges on Z.

- The computably randoms are the convergence points for all computable martingales.
- The Martin-Löf randoms that are density-one points are the convergence points for all left-c.e. martingales (Andrews, Cai, Diamondstone, Lempp, Miller; 2012).

The actual theorem

Theorem

Let $f\colon [0,1] \to \mathbb{R}$ be an interval-c.e. function. Let z be a convergence point for left-c.e. martingales. Then $f'(z)$ exists.

The basic connection:

- if f is interval-c.e., then $M(\sigma) = S_f([\sigma])$ is a left-c.e. martingale.
- Convergence of M on Z means that $\underset{\sim}{D}_2 f(z) = \widetilde{D}_2 f(z)$, i.e., f is dyadic differentiable at z.

The theorem says that we can get full differentiability for convergence points for left-c.e. martingales (but also looking at other left-c.e. martingales).

Recall: High dyadic slopes lemma

Suppose $f\colon [0,1] \to \mathbb{R}$ is a nondecreasing function. Suppose for a real $z \in [0,1]$ we have

$$\widetilde{D}_2 f(z) < p < \widetilde{D} f(z).$$

Let $\sigma^* \prec Z$ be any string such that $\forall \sigma\, [Z \succ \sigma \succeq \sigma^* \Rightarrow S_f([\sigma]) \leq p]$. Then the closed set

$$\mathcal{C} = [\sigma^*] - \bigcup \{(\sigma)\colon\ \sigma \succeq \sigma^* \ \wedge\ S_f([\sigma]) > p\},$$

which contains z, is porous at z.

Proposition

Let $f\colon [0,1] \to \mathbb{R}$ be interval-c.e. Then $\widetilde{D}_2 f(z) = \widetilde{D} f(z)$ for each non-porosity point z.

Proof.

Assume $\widetilde{D}_2 f(z) < \widetilde{D} f(z)$. Since f is interval c.e., the class $\mathcal{C}$ defined in the Lemma is effectively closed. This class is porous at z. Contradiction. □

Proof that $f'(z)$ exists for left-c.e. convergence points z

We may assume $z > 1/2$, else we work with $f(x + 1/2)$ instead of f.

- The real z is a a dyadic density one point, hence a (full) density-one point by the Khan-Miller Theorem.
- Then $z - 1/3$ is also a ML-random density-one point, so using the work of the Madison group discussed earlier, $z - 1/3$ is also a convergence point for left-c.e. martingales.
- In particular, both z and $z - 1/3$ are non-porosity points.

To complete the proof ...:

Let M be the martingale associated with the dyadic slopes of f.

- Note that M converges on z by hypothesis. Thus $\underset{\sim}{D}_2 f(z) = \widetilde{D}_2 f(z) = M(z)$.
- By the Proposition above we have $\widetilde{D}_2 f(z) = \widetilde{D} f(z)$.
- It remains to be shown that

  $$\underset{\sim}{D} f(z) = \underset{\sim}{D}_2 f(z).$$

  Since f is nondecreasing, $\underline{D} = \underset{\sim}{D}$ etc., so this will establish that $f'(z)$ exists.

Shifting by 1/3 yields the same dyadic derivative

Let $\widehat{f}(x) = f(x + 1/3)$, and let M' the martingale associated with the dyadic slopes of $\widehat{f}$.

Claim

$M(z) = M'(z - 1/3)$.

Proof.

Since $z - 1/3$ is a convergence point for c.e. martingales, M' converges on $z - 1/3$.

If $M(z) < M'(z - 1/3)$ then $\widetilde{D}_2 f(z) < \widetilde{D} f(z)$. However z is a non-porosity point, so this contradicts the Proposition.

If $M'(z - 1/3) < M(z)$ we argue similarly, using that $z - 1/3$ is a non-porosity point. □

Choosing some rational parameters

Assume for a contradiction that

$$\underset{\sim}{D}f(z) < \underset{\sim}{D}_2 f(z).$$

Then we can choose rationals p, q such that

$$\underset{\sim}{D}f(z) < p < q < M(z) = M'(z - 1/3).$$

Let $k \in \mathbb{N}$ be such that $p < q(1 - 2^{-k+1})$.

Let u, v be rationals such that

$$q < u < M(z) < v \text{ and } v - u \leq 2^{-k-3}(u - q).$$

Two Π^0_1 classes

Let $n^* \in \mathbb{N}$ be such that we have $S_f(A) \geq u$, for each $n \geq n^*$ and any interval A of length $\leq 2^{-n^*}$ that is basic dyadic or basic dyadic $+1/3$.

$$\begin{aligned} \mathcal{E} &= \{X \in 2^{\mathbb{N}} \colon \forall n \geq n^* M(X\restriction_n) \leq v\} \\ \mathcal{E}' &= \{W \in 2^{\mathbb{N}} \colon \forall n \geq n^* M'(W\restriction_n) \leq v\} \end{aligned}$$

- Let $0.Z$ be as usual the binary expansion of z. Let $0.Y$ be the binary expansion of $z - 1/3$.
- We have $Z \in \mathcal{E}$ and $Y \in \mathcal{E}'$.

We will show that $\mathcal{E}$ is porous at Z, or $\mathcal{E}'$ is porous at Y.

Low dyadic slopes for both types of intervals

Consider an interval $I \ni z$ of positive length $\leq 2^{-n^*-3}$ such that $S_f(I) \leq p$.

- Let n be such that $2^{-n+1} > |I| \geq 2^{-n}$.
- Let a_0 $[b_0]$ be least of the form $j2^{-n-k}$ $[j2^{-n-k} + 1/3]$, where $j \in \mathbb{Z}$, such that a_0 $[b_0] \geq \min(I)$.
- Let $a_v = a_0 + v2^{-n-k}$ and $b_v = b_0 + v2^{-n-k}$. Let r, s be greatest such that $a_r \leq \max(I)$ and $b_s \leq \max(I)$.

Since f is nondecreasing and $a_r - a_0 \geq |I| - 2^{-n-k+1} \geq (1 - 2^{-k+1})|I|$, we have $S_f(I) \geq S_f(a_0, a_r)(1 - 2^{-k+1})$, and therefore $S_f(a_0, a_r) < q$. (Slope at I is low, slope at $[a_0, a_r]$ can only be slightly larger.) Then there is an $i < r$ such that $S_f(a_i, a_{i+1}) < q$.
Similarly, there is $j < s$ such that $S_f(b_j, b_{j+1}) < q$.

Claim (Morayne-Solecki trick)

One of the following is true.

(i) z, a_i, a_{i+1} are all contained in a single interval taken from $\mathcal{D}_{n-3}$.

(ii) z, b_j, b_{j+1} are all contained in a single interval taken from $\mathcal{D}'_{n-3}$.

Proving porosity of one of the Π^0_1 classes

Let $\eta = Z\upharpoonright_{n-3}$ and $\eta' = Y\upharpoonright_{n-3}$.

If (i) holds for this I then there is α of length $k+3$ (where $[\eta\alpha] = [a_i, a_{i+1}]$) such that $M(\eta\alpha) < q$.

- So by the choice of $q < u < v$ and since $M(\eta) \geq u$ there is β of length $k+3$ such that $M(\eta\beta) > r$.
- This yields a hole in $\mathcal{E}$, large and near $z = 0.Z$ on the scale of I, which is required for porosity of $\mathcal{E}$ at Z.

Similarly, if (ii) holds for this I, then there is α of length $k+3$ (where $[\eta'\alpha] = [b_j, b_{j+1}]$) such that $M'(\eta'\alpha) < q$. This yields a hole large and near $z - 1/3 = 0.Y$ on the scale of I required for porosity of $\mathcal{E}'$ at Y.

Thus, if case (i) applies for arbitrarily short intervals I, then $\mathcal{E}$ is porous at Z, whence z is a porosity point. Otherwise (ii) applies for intervals below a certain length. Then $\mathcal{E}'$ is porous at Y, whence $z - 1/3$ is a porosity point. Either case is a contradiction.

Some open questions

Question

Study effective analogs of Rademacher's theorem that every Lipschitz function on $\mathbb{R}^n$ is a.e. differentiable.

Question

How much randomness is needed to ensure differentiability of interval-Π^1_1 functions?

Chong, N and Yu have shown that each Π^1_1 random is a density-one point for Σ^1_1 classes. Maybe this latter property does it, by analogy with the computable case.

Question

If Z is a ML-random density-one point, is it Oberwolfach random? Equivalently, does it fail to compute some K-trivial?

Full proofs of the two theorems are on the 2013 Logic blog, available on my web site.