Extensions of Levy-Shoenfield Absoluteness

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Levy-Shoenfield Absoluteness

Let $\mathcal{P}(x)$ be a lightface $\Sigma_1$predicate.

Theorem (Levy, Shoenfield) $\exists x \mathcal{P}(x) \rightarrow [L \vDash \exists x \mathcal{P}(x)]$.

The proof works when $\mathcal{P}(x)$ has a parameter in $L(\omega_1^L)$,
but can fail dramatically
when $\mathcal{P}(x)$ has a parameter $\gamma \in L - L(\omega_1^L)$.

Cones

Let κ be a L-cardinal, and v a bounded subset of κ.

A set of the form

$$\{w \mid \exists\delta < \kappa \wedge v \in L(\delta, w))\}.$$

is called a κ-**cone** with vertex v.

Note: Cones belong to V

Amenability

Let λ be a limit ordinal and v an unbounded subset of λ.

v is **amenable** iff $\forall\gamma < \lambda \;\; (v \cap \gamma) \in L(\lambda)$.

Theorem A Let $\mathcal{Q}(x)$ be a Δ_0^{ZF} predicate whose sole parameter is an ordinal c.
Suppose v is an amenable solution of $\mathcal{Q}(x)$, $\lambda = \sup v$, and κ is the least L-cardinal greater than c and λ.

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Suppose v is an amenable solution of $\mathcal{Q}(x)$, $\;\lambda = \sup v$, and κ is the least L-cardinal greater than c and λ.

Assume the κ-cone with vertex v is a set of solutions of $\mathcal{Q}(x)$. Then $\mathcal{Q}(x)$ has a solution in L.

E-Recursion

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Moschovakis (1967): "$\{e\}(x)$ diverges" is a Σ_1^{ZF} predicate.

Divergence witnesses in L

Theorem B Let $c \in L$. Suppose $\exists$ amenable v such that $\{e\}(v, c)$ diverges, $\lambda = \sup v$, and κ is the least L-cardinal greater than c and λ.

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Assume the κ-cone with vertex v is a set of solutions of "$\{e\}(x, c)$ diverges."

Then $\{e\}(x, c)$ diverges for some $x \in L$.

Proof of Theorem B

The argument "takes place" in L.

Let $m = \max\{c, \lambda\}$ and $\rho =$ cofinality of λ.
$H_1(x)$ denotes the standard Σ_1 hull of x.
Thus $x \subseteq H_1(x) \preceq_1 L$.

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Define H^δ for $\delta \leq \rho$.

$$H^0 = H_1(m \cup \{m, \kappa\}), \;\; H^{\delta+1} = H_1(H^\delta \cup \{H^\delta\}),$$

and $H^\gamma = \cup\{H^\delta \mid \delta < \gamma\}$ when γ is a limit.

Proof II

Collapsing $\mathbf{H}^\rho$

Define $\overline{H^\rho} = t(H^\rho)$. t is the collapsing map.
$t(\kappa)$ is the greatest cardinal in the sense of $\overline{H^\rho}$.
$\overline{H^\rho}$ thinks $t(\kappa)$ is regular.

Let $gc(\kappa)$ denote $t(\kappa)$.
$\mathit{cofinality}(gc(\kappa)) = \rho$ (in L).

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Let $gc(\kappa)$ denote $t(\kappa)$.
$cofinality(gc(\kappa)) = \rho$ (in L).
$\Sigma_2^{\overline{H^{\rho}}}$ cofinality of $gc(\kappa)$ equals $\Sigma_2^{\overline{H^{\rho}}}$ cofinality of $ord(\overline{H^{\rho}}) = \rho$.
$t(c) = c$. $t(\lambda) = \lambda$. $t(m) = m$.

Proof III

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There is a Σ_2^J map of $gc(\kappa)$ onto J.
The Σ_2^J cofinality of $gc(\kappa)$ is $gc(\kappa)$.

G denotes a subset of $gc(\kappa)$ generic over J.
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Generic G's can be built in L in $gc(\kappa)$-many steps.

A Generic G in L

Let $G_0 \in L$ be generic over J

If $\{e\}(G_0, c)$ diverges., then all is well.

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Suppose $\{e\}(G_0, c)$ converges.

Then the computation is in J.

$\exists p \;\; p \Vdash [\{e\}(\mathcal{G}, c) \text{ converges}] \;\; G_0 \in p$.

A Generic G Outside L

There exists a G_1 generic over J
such that $v \in L(\kappa, G_1)$ and $p \in G_1$.
(v is coded into G_1 using the amenability of v.)

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Thus $G_1 \in \kappa$-cone with vertex v. So $\{e\}(G_1, c)$ diverges.

But $p \Vdash [\{e\}(\mathcal{G}, c)$ converges]. So $\{e\}(G_1, c)$ converges.

So $\{e\}(G_0, c)$ diverges.

Gödel Condensation

Let H be a Σ_n substructure of L.

The condensing map $m : H \longrightarrow m[H]$ is defined by

$$m(x) = \{m(y) \mid y \in x\}.$$

Theorem (Gödel) $m[H]$ is an initial segment of L.

Infinitary Logic Review I

$\mathcal{L}$ is a set of relation symbols, function symbols and individual constants.

$\mathcal{L}_{\infty,\omega}$ is the set of well formed formulas.

conjunctions and disjunctions are of arbitrary length.

quantifier prefixes are of finite length.

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A **fragment** Z is a set of wffs such that

Z is closed under finitary formation rules and

every subformula of a formula of Z is a formula of Z.

Infinitary Logic Review II

The axioms and rules of infinitary logic

are the same as those of first order logic

except for one infinitary rule:

a deduction of $\mathcal{F}_i$ for each $i \in I$

constitutes a deduction of $\wedge\{\mathcal{F}_i \mid i \in I\}$.

Let Z be a fragment. Suppose $T \subseteq Z$.

T is **complete** iff

$\mathcal{F} \in T$ or $(\neg\mathcal{F}) \in T$ for each $\mathcal{F} \in Z$.

and $\vee\{\mathcal{F}_i \mid i \in I\} \in T$

implies $(\exists i \in I)[\mathcal{F}_i \in T]$

Infinitary Logic Review III

Suppose Z is a fragment and $T \subseteq Z$.

Fact 1 If T is countable and consistent, then T has a model.

Definition T is **finitarily consistent** iff no finite deduction from T yields a contradiction.

Fact 2 If T is finitarily consistent and complete, then T is consistent.

Fact 3 If T is countable, finitarily consistent, and complete, then T has a model.

Condensing A Theory

Let $\mathcal{L}$ be countable and T a consistent, complete theory in some fragment Z of $\mathcal{L}_{\infty,\omega}$.

H is a Σ_1 substructure of V. Assume $T \in H$.

$m : H \longrightarrow m[H]$ is the condensing map.

Assume $m(Z) \subseteq Z$.

Theorem A $m(T)$ is an initial segment of T.

Proof of Theory Condensation

Construct $\mathcal{A}$, a countable partial model of $T \cap H$.

$T \cap H$ is finitely consistent and complete

in the "fragment" $Z \cap H$.

If $\mathcal{A} \models \vee\{\mathcal{F}_i \mid i \in I\}$,

then $\exists i\ [\mathcal{A} \models \mathcal{F}_i$ and $\mathcal{F}_i \in H]$.

Let $\mathcal{F} \in \mathit{ElemDiag}(\mathcal{A})$.

Proposition $\mathcal{A} \models \mathcal{F} \longrightarrow \mathcal{A} \models m(\mathcal{F})$. (by induction on $\mathcal{F}$)

Let $\mathcal{F} \in (T \cap H)$. Then $\mathcal{A} \models m(\mathcal{F})$;

$m(\mathcal{F}) \in \mathit{Lang}(T)$, so $m(\mathcal{F}) \in T$.

Mild Stability

Let T be an uncountable, consistent, complete theory in some fragment of $\mathcal{L}_{\infty,\omega}$.

Definition T' is a **countable condensate** of T iff
$\exists H$, a countable Σ_1 substructure of V,
such that $T \in H$ and $m(T) = T'$.

Let $S(T')$ be the set of all n-types of T' $(n \geq 1)$ defined syntactically.

Definition T is **mildly stable** iff
for every countable condensate T' of T,
$S(T')$ is countable.

Absoluteness Of Mild Stability

Let $A_1(x)$ be the least Σ_1 admissible set with
x as a member.

Theorem B T is mildly stable iff
$ST \in A_1(T)$.

$\mathbf{ST \in A_1(T)}$ means:

$\exists W \in A_1(T)$

W is a non-empty set of n-types of T

$T \vdash (\forall \overrightarrow{x}) \vee \{p(\overrightarrow{x}) \mid p(\overrightarrow{x}) \in W\}$.

Existence Of Types

Let T be an uncountable, consistent, complete theory in some fragment of $\mathcal{L}_{\infty,\omega}$.

Let ST be the set of all n-types of T.

Theorem C

Assume T is mildly stable and $(\exists x)\mathcal{F}(x) \in T$.

Then $\exists\ p \in ST$ such that $\mathcal{F}(x) \in p$.

Proof Of Existence Of Types

Let H be a Σ_1 substructure of V such that $T \in H$.

Let $m : H \longrightarrow m[H]$ be the condensing map.

$m(T)$ has a countable model $\mathcal{A}$. $\mathcal{A} \models m(\exists x \mathcal{F}(x))$.

For some $a \in A$, $\mathcal{A} \models m(\mathcal{F}(a))$.

Define $q = \{m(\mathcal{G}(x)) \mid \mathcal{A} \models m(\mathcal{G}(a))\}$.

$S(m(T)) \in m(H)$ by moderate stability of T.

So $q \in m(H)$ and $p = m^{-1}(q) \in ST$.

Moderate Stability

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Let $\mathcal{L}$ be a countable first order language. Assume $V = L(\mathcal{L})$.
Let $T_{\omega_2} \subseteq \mathcal{L}_{\omega_2,\omega}$ be consistent and complete.
$T^{'}$ denotes a countable condensate of T_{ω_2}. Suppose $T^{'}$ has a countable model A whose n-tuples realize atoms of T_{ω_2}.

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Definition T_{ω_2} is **moderately stable** iff T_{ω_2} is mildly stable and for every T' as above, $T_{\omega_2} \cup Dia(A)$ has a consistent complete extension to $\mathcal{L}_{A,\omega_2,\omega}$.

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Theorem If T_{ω_2} is moderately stable,
then T_{ω_2} has a model of size ω_2.