C.E. equivalence relations under computable reducibility

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CCR, Buenos Aires Semester

May 15, 2013

Opening: some initial references (1)

- This is joint work with Uri Andrews, Steffen Lempp, Joseph S. Miller, Keng Meng Ng, and Andrea Sorbi: *Universal computable enumerable equivalence relations*, to appear (I will refer to this paper as A)

Various aspects of the theory of computable enumberable relations (ceers) have been studied extensively. One of the reasons for such interest lies in the fact that ceers can be approached in many different ways. Here is a list (of course non-exhaustive) of these possible approaches:

- [Ershov, 1977] gave the first definition of our reducibility in the context of the theory of numberings, "in order to study some recursion-theoretic concepts from a *global* point of view";

Opening: some initial references (2)

- [Visser, 1980], [Montagna, 1982], [Bernardi and Sorbi, 1983], were motivated, at least in part, by the study of provable equivalence among formal systems;
- [Lachlan, 1987] investigated ceers considering computable isomorphism types;
- more recently, new motivations occured with the formulation of a hierarchy of relative complexity among different c.e. classification problems (a computable analogue of the so-called Borel reducibility), see [Gao and Gerdes, 2001], [Coskey, Hamkins and Miller, 2012]

Here, our focus is mainly on the degree structure generated by Ershov reducibility.

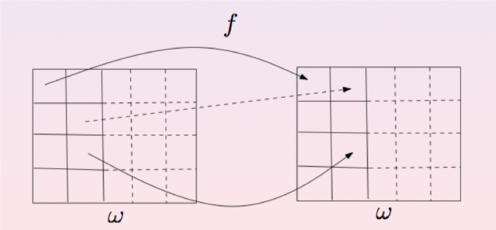
Object of the study

We study ceers under the following reducibility:

Definition

$R \leq S$ if there exist a computable funcion f such that $x \, R \, y \Leftrightarrow f(x) \, S \, f(y)$.

It is immediate to see that this reducibility is 1-1 on equivalence classes:



Degrees and universality

Degrees are introduced in the natural way:

Definition

Let $R \equiv S$ if $R \leq S$ and $S \leq R$. Denote by $deg(R)$ the $\equiv$-equivalence classe of R and define

$$deg(R) \leq deg(S) \Leftrightarrow R \leq S.$$

Let $\mathcal{P}$ denotes the poset of degrees of ceers.

Definition

A ceer S in *universal* if $R \leq S$ for any ceer R.

The category of ceers

Sometimes it would be convenient to view ceers as objects of the following category, Eq^P:

- Objects of Eq^P: c.e. equivalence relations on ω;
- Morphism from R to S: functions $\mu : \omega/_R \to \omega/_S$ s.t. there is a computable function f for which $\mu([x]_R) = [(f(x)]_S$

Some questions

1. What can we say about the algebraic structure of $\mathcal{P}$? For instance, is it a lattice?
2. Is the first-order theory of $\mathcal{P}$ undecidable?
3. Are there universal ceers?
4. What is the best characterization that we can find for the set of universal ceers?
5. Is there a unique notion of universality among ceers?

Indices for ceers

Given a c.e. set A, we denote with A^* the equivalence relation generated by A (here we view A as coding a binary relation via Cantor Pairing function). Then, let $R_e = W_e^*$. It is clear that every c.e. equivalence relation occurs (infinitely many times) in this enumeration.

We call this enumeration the *canonical enumeration* of the set of ceers.

Outline

This presentation is divided in two parts.

1. In the first part, some aspects of the structure of $\mathcal{P}$ are under examination. We will give a couple of results on the general structure of $\mathcal{P}$, looking to a particular class of low-complexity ceers;
2. then, we move to the degree of universal ceers.

The structure of $\mathcal{P}$

Computable equivalence relations (1)

Computable equivalence relations are the simplest ones, w.r.t to our reducibility.

Definition

Id_n denote the following ceer with n equivalence classes:

$$x \, Id_n \, y \Leftrightarrow x \equiv y \ (mod \ n)$$

Definition

Id denote the identity relation on ω.

Computable equivalence relations (2)

Lemma

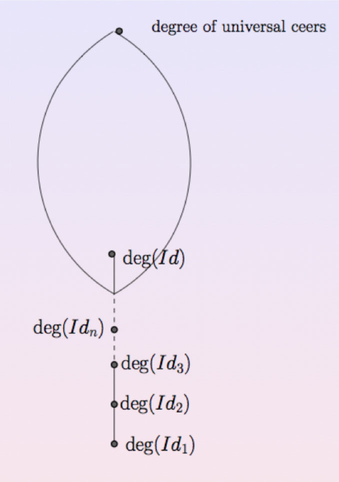
R is a computable equivalence relation iff $R \leq Id$.

These facts, easy to prove, give us a full characterization of computable equivalence relations under our reducibility (for more see [Gao, Gerdes]):

- $\mathcal{P}$ is a bounded poset: The least element is $deg(Id_1)$; the greatest element is given by the degree of universal ceers;
- $\mathcal{P}$ has a linearly ordered initial segment, $\mathcal{I}_\omega$: $deg(Id_1) < ... < deg(Id_n) < ... < deg(Id)$;
- every ceer with n equivalence classes lies in $deg(Id_n)$;
- $deg(Id)$ consists of all computable equivalence relations with infinitely many equivalence classes.

It is maybe important to notice that $\mathcal{P}$ got infinitely many computable degrees.

A first picture of $\mathcal{P}$



For the rest of the talk, we will consider ceers with infinitely many equivalence classes.

A significant fragment of $\mathcal{P}$: unidimensional ceers

Definition (Gao, Gerdes)

A ceer R is *unidimensional* if there exists a c.e. set A s.t.

$$x \, R \, y \Leftrightarrow x = y \vee (x, y \in A).$$

We use the following notation for unidimensional ceers: R_A.

Lemma

R_K is universal among unidimensional ceers, where K is the halting set.

Proof.

We will shortly prove something more general. □

Is *Id* the minimum among non-computable ceers?

Lemma (A; Gao, Gerdes)

Id is not reducible to all non-computable ceers.

Proof.

Let S be a simple set. If $Id \leq S$ via f, then f is 1-1 and there is at most one i s.t. $f(i) \in S$. So, either $f[\omega] \subseteq \omega - S$ or $f[\omega - \{i\}] \subseteq \omega - S$. In either case, $\omega - S$ contains an infinite c.e. set. □

Unidimensional ceers and 1-reducibility (1)

Unidimensional ceers are natural ways to represent c.e. sets as ceers. It is natural to ask whether they reflect the same structure of some reducibility among sets.

The following lemma give us a positive answer.

Lemma (A; Hamkins, Miller, Coskey)

Let A, B be infinite c.e. sets, then $R_A \leq R_B \Leftrightarrow A \leq_1 B$.

Proof.

($\Leftarrow$) is obvious because: the same function f that $1-$reduces A to B also reduces R_A to R_B.

($\Rightarrow$): on board. □

Unidimensional ceers and 1-reducibility (2)

Lemma (A)

If $Id \leq R \leq R_A$ then there exists a c.e. set B s.t. $R \equiv R_B$.

Proof.

If $Id \leq R \leq R_A$, and $R \leq R_A$ via a computable function f, then the $range(f)$ is c.e. infinite set, thus computably isomorphic to ω. Let $g : range(f) \to \omega$ be a computable bijection and take $B = g[A \cap range(f)]$. Then $R_B \leq R$ via h where

$$h(x) = \mu y.[g(f(y)) = x]$$

and $R \leq R_B$ via $g \circ f$. □

A deeper look at $\mathcal{P}$

The combination of the last two lemmas has noteworthy consequences on $\mathcal{P}$.

Corollary (A)

The interval $[deg(Id), deg(R_K)]$ *is isomorphic to the interval of c.e.* $1-$*degrees* $[\mathbf{0}_1, \mathbf{0}^{'}]$.

Thus, we can export results from the 1-degrees to $\mathcal{P}$.

Corollary (A)

$\mathcal{P}$ *is neither an upper semilattice nor a lower semilattice.*

Proof.

It follows from the fact the $[\mathbf{0}_1, \mathbf{0}^{'}]$ is neither an upper semilattice nor a lower semilattice (see [Young, 1963]). □

Undecidability of $\mathcal{P}$

Theorem (A)

The first order theory of $\mathcal{P}$ is undecidable.

Proof.

The proof makes use of Nies Transfer Lemma (see [Nies, 1996]). The idea of the proof is the following: we know (by [Lachlan, 1969]) that the topped initial segments of $[\mathbf{0}_1, \mathbf{0}']$ are exactly the finite distributive lattices, thus the same is true for the interval $[deg(Id), deg(R_K)]$ in $\mathcal{P}$. Then we can define the first order theory of finite distributive lattice in $\mathcal{P}$, undecidability follows from the fact that this theory is hereditary undecidable. $\square$

Universal ceers

The degree of universal ceers

The degree of universal ceers is not empty, in fact:

Definition

Let U be defined by:

$$\langle x, z\rangle U \langle y, z\rangle \Leftrightarrow x R_z y.$$

Fact

U is universal.

Two questions on universal ceers

Question

In analogy to the classic case, two questions arise:

1. Does an equivalent of Myhill Theorem stand for ceers? (i.e., Are any two ceers in $deg(U)$ computably isomorphic?)
2. In this context what is the link beetwen universality and effective inseparability?

Precomplete ceers

Definition (Ershov; Malt'sev)

A ceer R is *precomplete* if R has infinitely many equivalence classes and for every partial computable function φ there exists a total computable function f s.t. for all n,

$$\varphi(n) \downarrow \Rightarrow \varphi(n) \, R \, f(n).$$

Example (Visser)

Let g_n be an effective coding of the Σ_n−sentences of any "sufficiently strong" and consistent first-order theory T (e.g., $T = PA$) and let $\sim_n$ defined by

$$x \sim_n y \Leftrightarrow T \vdash g_n^{-1}(x) \leftrightarrow g_n^{-1}(y).$$

Then $\sim_n$ is precomplete.

An alternative characterization of precomplete ceers

The following will be helpful in showing the failure of Myhill Isomorphism Theorem in the context of ceers.

Definition

A partial computable function Δ is called a *diagonal* function for a ceer R, if for every x s.t. $\triangle(x)\downarrow$, we have that $\neg(\triangle(x)\, R\, x)$.

Theorem (Ershov Recursion Theorem)

R is precomplete iff there is a computable funcion fix s.t., for every n,

$$\varphi_n(fix(n))\downarrow \Rightarrow \varphi_n(fix(n))\, R\, fix(n)$$

Corollary

A precomplete ceer R cannot admit a total computable function as a diagonal function.

Anti Diagonal Normalization Theorem

Theorem (Visser)

Let R be a precomplete ceer, and let Δ be a diagonal function for R. Then for every partial computable function φ, there exists (uniformly from φ and Δ) a total computable function g s.t., for every x,

$* \; \varphi(x)\downarrow \Rightarrow \varphi(x) \, R \, g(x)$;

$* \; \varphi(x)\uparrow \Rightarrow g(x) \notin dom(\Delta)$.

Precomplete ceers vs effective inseparability

Theorem

[Bernardi and Sorbi] Every precomplete ceer S yields to a partition of ω in effective inseparable sets.

Proof.

Take $a, b \in \omega$ s.t. $\neg(a\, S\, b)$. Then, for W_y and W_z disjoint c.e. sets, define Δ and ψ as follows:

$$\Delta(x) \begin{cases} b & xSa \\ a & xSb \\ \uparrow & o.w. \end{cases} \qquad \psi(x) \begin{cases} a & x \in W_y \\ b & y \in W_z \\ \uparrow & o.w. \end{cases}$$

It is clear that Δ is diagonal for S. By the ADN-Theorem there exists a total computable function g (uniformly, given an index of ψ) s.t.

$x \in W_y \Rightarrow g(x)\, S\, a \Rightarrow g(x) \in [a]_S$;

$x \in W_z \Rightarrow g(x)\, S\, b \Rightarrow g(x) \in [b]_S$;

$x \notin W_y \cup W_z \Rightarrow g(x) \notin dom(\Delta) = [a]_S \cup [b]_S$.

In conclusion, (W_y, W_z) reduces to $([a]_S, [b]_S)$.

Two theorems on precomplete ceers

Using ADN Theorem and the fact that every precomplete ceer is e.i. it is possible to prove the following:

Theorem (Bernardi and Sorbi)

Every precomplete ceer is universal.

Theorem (Lachlan)

All precomplete ceer are isomorphic.

Corollary

All precomplete ceers are all isomorphic to $\sim_1$.

Failure of Myhill Theorem

Theorem (Bernardi and Sorbi)

$\sim_T$ is not precomplete, but universal.

Proof.

Consider the following computable function $N(x) = g(\neg g^{-1}(x))$. Then, if there exists x s.t. $N(x) \sim_T x$ it follows that $T \vdash \neg(g^{-1}(x)) \leftrightarrow g^{-1}(x)$ and this is not possible by consistency of T. So, N is total diagonal function for $\sim_T$.

The universality of $\sim_T$ immediatly follows from the fact that $\sim_n \leq \sim_T$ for every n. □

Corollary

There are universal not computably isomorphic ceers.

U.f.p. ceers (1)

Althought not precomplete, $\sim_T$ is "locally" precomplete: if we take a computable function φ with finite range, all sentences in *range*(φ) are Σ_n for some n, and thus we can use $\sim_n$ to obtain a total f for φ. This leads to the following definition:

Definition

R is *uniformly finitely precomplete* (u.f.p.) if for every partial computable function φ and finite set D there uniformly exists a total computable function f such that, for all n,

$$\varphi(n)\downarrow\in [D]_R \Rightarrow \varphi(n)\, R\, f(n)$$

U.f.p. ceers (2)

Fact

Every precomplete is u.f.p., and T is u.f.p.

Indeed, given φ and D, all the senteces in D fall into some finite level Σ_n.

The following holds:

Theorem (Montagna)

Every u.f.p. ceer is universal.

e-complete

So inside the class of u.f.p. ceers we got (almost) two different isomorphism types: precomplete ceers, for which stand Ershov Recursion Theorem, and the ceers that have a total diagonal function (e.g. T).

Definition (Montagna; Lachlan)

A ceer R is e-complete if it is u.f.p. and it has a total diagonal function.

T, of course, is e-complete.

Theorem (Montagna)

Every e-complete ceers are isomorphic to T.

Effective inseparability for ceers

Definition

A ceer R is:

$*$ *effective inseparable* if it yields to a partition of ω in e.i. sets;

$*$ *uniformly effective inseparable* (u.e.i) if it is e.i. and there is a computable function g(a,b) such that if $[a]_R \cap [b]_R = \emptyset$; then $\varphi_{g(a,b)}(u, v)$ is a productive function for the pair $[a]_R$, $[b]_R$.

U.f.p. vs u.e.i.

Theorem (Smullyan)

Every pair of e.i. sets is m-complete.

Thus, it is natural to study the interplay between effective inseparability and universality for ceers. We already proved that every precomplete ceer is e.i.; the same proof (by means of some modifcations) can be arranged to obtain the following:

Theorem

Every u.f.p. ceer is u.e.i.

Question

Does e.i. imply universality for ceers?

Every u.e.i. ceer is universal

The theorem below subsumes all universality results seen so far:

Theorem (A)

Every u.e.i. ceer is universal.

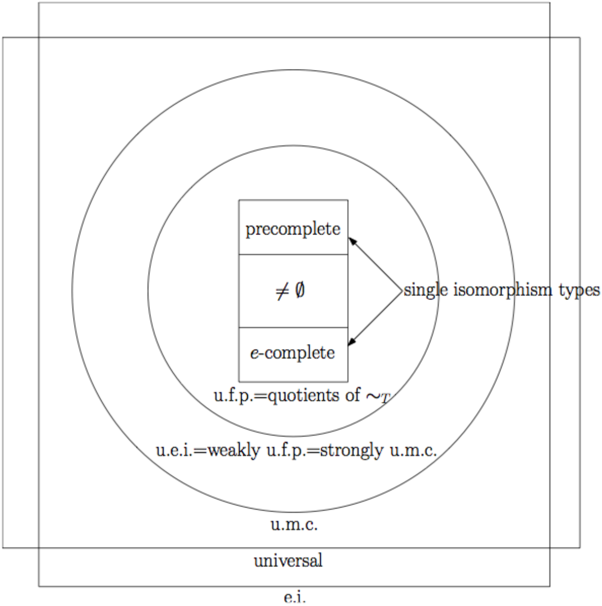
The proof is very long and challenging. It introduces two more classes of ceers: weakly u.f.p. and strongly u.m.c. and proves the equivalence of the three. Then universality is given by the definition of strongly u.m.c ceers.

Uniformity, however, cannot be discarded, in fact:

Theorem (A)

There exists a e.i. ceer that is not universal.

A final picture of the degree of universal ceer (see [A])



Further directions

1. Universal ceers are better understood than non-universal. [Gao and Gerdes] is the best reference for these, but an abudance of questions remain open.
2. Does the class of u.e.i ceers coincide with the class of u.f.p. ceers?
3. Algebraic espressivity of ceers.
4. We studied **c.e.** equivalence relations under **computable** reducibility. Every (meaningful) substitution of the boldface terms may lead to something interesting.

References

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Thank you

(and sorry for my weird pronunciation!)