SJT as an analog of K-triviality

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Figuiera, Nies and Stephan defined Strong Jump Traceability (SJT) in the hopes of characterizing K-triviality.

Definition

An *order* is a function $h : \omega \to \omega$ which is positive, non-decreasing and has infinite limit.

Definition

If f is a partial function and h is an order, an *h-trace* for f is a uniformly given sequence of c.e. sets $\langle T_e \rangle_{e \in \omega}$ with $|T_e| \leq h(e)$, and $f(e) \in T_e$ for all e in the domain of f.

Definition

A set A is *Strongly Jump Traceable* if every A-computable partial function has an h-trace for every computable order h.

K-trivial:

- The K-trivials form an ideal within Δ_2^0.
- This ideal is generated by its c.e. elements.
- K-triviality is characterized by obeying all additive cost functions.

SJT:

- The SJT form an ideal properly contained in the K-trivials. [Cholak, Downey, Greenberg]
- This ideal is generated by its c.e. elements. [Diamondstone, Greenberg, T]
- SJT is characterized by obeying all benign cost functions. [Greenberg, Nies; Greenberg, Hirschfeldt, Nies; Diamondstone, Greenberg, T]

K-trivial:

- Every K-trivial is computable from a difference random.
- Every c.e. set computable from a difference random is K-trivial.
- Every set which is a base for difference randomness is K-trivial (and conversely).

SJT:

- Every SJT is computable from a Demuth random. [Greenberg, T]
- Every c.e. set computable from a Demuth random is SJT. [Kučera, Nies]
- Every set which is a base for Demuth randomness is SJT, but the converse fails. [Nies; Greenberg, T]
- Every set which is a base for $\text{Demuth}_{\text{BLR}}$ randomness is SJT (and conversely). [Nies; Bienvenu, Downey, Greenberg, Nies, T; Greenberg, T]

K-trivial:

- Every MLR which is not OWR computes every K-trivial.
- There is a Δ^0_2 difference random which computes every K-trivial.

SJT:

- Every MLR which is not weak-Demuth random computes every SJT. [?]
- There is no Δ^0_2 Demuth random which computes every SJT. [Ng]

Definition

A *Demuth test* is a sequence $\langle V_n \rangle_{n\in\omega}$ of open sets with $\lambda(V_n) \leq 2^{-n}$, such that there is an ω-c.a. function giving Σ^0_1-indices for the V_n.

A real *passes* a Demuth test if it is contained in only finitely many of the V_n. (Note that the V_n need not be nested.)

A real is *Demuth random* if it passes every Demuth test.

A Demuth test is like a non-nested Martin-Löf test except you can empty a test element and begin again a computably-bounded number of times.

Proposition

There is no Δ^0_2 Demuth random which computes every SJT.

Proof.

If X were a Demuth random which computes every SJT, $\{e : W_e \leq_T X\} = \{e : W_e \text{ is SJT}\}$.

Ng showed that $\{e : W_e \text{ is SJT}\}$ is Π^0_4-complete.

$\{e : W_e \leq_T X\}$ is $\Sigma^0_3(X)$.

Every Demuth random is GL_1. So a Δ^0_2 Demuth random is low.

If X is low, $\Sigma^0_3(X) = \Sigma^0_3$. □

Definition

A *weak Demuth test* is a Demuth test for which the V_n are nested.

A real is *weak Demuth random* if it passes every weak Demuth test.

Definition

A (monotonic, limit-condition) *cost function* is a function $c : \omega \times \omega \to \mathbb{Q}$ satisfying:

- $c(x+1, s) \leq c(x, s) \leq c(x, s+1)$;
- $c(x) = \lim_s c(x, s) < \infty$; and
- $\lim_x c(x) = 0$.

For a cost function c, and a fixed n, consider the following sequence:

- $\ell_0^n = 0$;
- $\ell_{i+1}^n = (\mu s > \ell_i^n)[c(\ell_i^n, s) > 2^{-n}]$.

Since $\lim_x c(x) = 0$, this sequence is finite.

A cost function is *g-benign* if $g(n)$ bounds the length of $\langle \ell_0^n, \ell_1^n, \ldots \rangle$ for all n. A cost function is *benign* if it is g-benign for some computable function g.

Definition

If c is a cost function and $\langle A_s \rangle_{s\in\omega}$ is a sequence of sets, let $c(A_s) = c(y_s, s)$, where y_s is the least element of $A_s \triangle A_{s+1}$.

$\langle A_s \rangle_{s\in\omega}$ *obeys* c if $\sum c(A_s) < \infty$.

If A is a Δ^0_2 set, A *obeys* c if it has some computable approximation $\langle A_s \rangle_{s\in\omega}$ which obeys c.

Theorem (Greenberg, Nies; Greenberg, Hirschfeldt, Nies)

If A obeys all benign cost functions, then A is SJT.

Theorem (Diamondstone, Greenberg, T)

Every SJT obeys all benign cost functions.

All K-trivials obey all additive cost functions.

In fact, there is a single additive cost function c_K such that obeying c_K implies K-triviality.

This implies that every K-trivial is computable from a c.e. K-trivial, via the *change set*:

If A is K-trivial, let $\langle A_s \rangle_{s\in\omega}$ witness that A obeys c_K. Then let $B = \{(x, n) : (\exists s_1 < s_2 < \cdots < s_n)(\forall i \leq n)[x \in A_{s_i} \triangle A_{s_i+1}]\}$. Assume $\#(x, n) \geq x$. Then B is c.e., $A \leq_T B$, and B has a natural approximation $\langle B_s \rangle_{s\in\omega}$ with $c_K(B_s) \leq c_K(A_s)$ for all s.

This same argument does not suffice for SJT.

There is no single benign cost function which characterizes SJT.

Different cost functions require different witnessing approximations $\langle A_s \rangle_{s\in\omega}$, which generate different change sets.

Theorem (Diamondstone, Greenberg, T)

There is a benign cost function $\dot{c}$ such that if A obeys $\dot{c}$, there is a c.e. $B \geq_T A$ which obeys every cost function which A does.

Corollary

Every SJT is computable from a c.e. SJT.

We want to build $\dot{c}$ by studying the witnessing approximation $\langle A_s \rangle_{s\in\omega}$. This is not as impossible as it sounds.

Let $\left\langle \langle A_s^n \rangle_{s\in\omega} \right\rangle_{n\in\omega}$ be a listing of all partial computable sequences of sets. We will build cost functions $\langle c_n \rangle_{n\in\omega}$. c_n will be built under the assumption that $\langle A_s^n \rangle_{s\in\omega}$ is total and obeys c_n with total cost at most 2^n. Even if this assumption fails, we must meet the following:

- c_n is total;
- $(\forall x, s) c_n(x, s) \leq 1$;
- The $\langle c_n \rangle_{n\in\omega}$ are uniformly benign.

Then we set $\dot{c} = \sum 2^{-n} c_n$, and $\dot{c}$ will be a benign cost function.

Suppose A obeys $\dot{c}$. Then there is some approximation $\langle A_s \rangle_{s\in\omega}$ to A with $\sum \dot{c}(A_s) \leq 1$.

There is some k with $\langle A_s^k \rangle_{s\in\omega} = \langle A_s \rangle_{s\in\omega}$.

$\sum c_k(A_s^k) \leq \sum 2^k \dot{c}(A_s) \leq 2^k$. So c_k is correct about its assumption.

Now we focus on building c_k, but we drop the k. So we have a partial computable sequence of sets $\langle A_s \rangle_{s\in\omega}$ and a constant 2^k. We need to build a total benign cost function c which is bounded by 1. If $\langle A_s \rangle_{s\in\omega}$ is total and $\sum c(A_s) \leq 2^k$, we need to construct a B which obeys all cost functions which A obeys.

We need to study all cost functions, and watch if A obeys them. That means guessing at approximations which might obey them. Fortunately, we can restrict our attention to a certain kind of approximation:

Lemma

If d is a cost function, A obeys d, and $\langle A_s \rangle_{s\in\omega}$ is any computable approximation to A, then there is a computable, strictly increasing function g with $\left\langle A_{g(s)} \right\rangle_{s\in\omega}$ obeying d with total cost at most 1.

So let $\left\langle (d^i, g^i) \right\rangle_{i\in\omega}$ be a listing of cost functions paired with partial strictly increasing functions.

First idea:

Let B be the change set for $\langle A_s \rangle_{s\in\omega}$ and let $\langle B_s \rangle_{s\in\omega}$ be the natural approximation. If g^i is total and $\sum d^i(A_{g^i(s)}) < \infty$, we will make $\sum d^i(B_{g^i(s)}) < \infty$.

At stage s, suppose m is the largest element of the domain of g^i so far, with $g^i(m) < s$. We see $A_s(z) \neq A_{g^i(m)}(z)$. This change will be recorded in the change set, and $\sum d^i(B_{g^i(s)})$ will pay at most $d^i(z, m)$ for it.

There are two possibilities:

1. If $A(z)$ does not change from s to $g^i(m+1)$, then $A_{g^i(m+1)}(z) \neq A_{g^i(m)}(z)$, and so $\sum d^i(A_{g^i(s)})$ must pay $d^i(z, m)$ for it.
2. If $A(z)$ does change, then for some $t \geq s$, $A_{t+1}(z) \neq A_t(z)$, and so $\sum c(A_s)$ must pay at least $c(z, s)$ for it.

Without knowing which of the two possibilities it is, we can set $c(z, s) = d^i(z, m)$. Then one of the two sums will pay at least $d^i(z, m)$, and so $\sum d^i(A_{g^i(s)}) \leq \sum d^i(A_{g^i(s)}) + \sum c(A_s) < \infty$.

Unfortunately, we need to make c total even if $\langle A_s \rangle_{s \in \omega}$ is not. So we can't wait for $A_s(z)$ to converge before defining $c(z, s)$.

Second idea:

We will build a function f. If $\langle A_s \rangle_{s\in\omega}$ is total, f will be total.

Let B be the change set for $\left\langle A_{f(s)} \right\rangle_{s\in\omega}$ and let $\langle B_s \rangle_{s\in\omega}$ be the natural approximation. If g^i is total and $\sum d^i(A_{f\circ g^i(s)}) < \infty$, we will make $\sum d^i(B_{g^i(s)}) < \infty$.

At stage s, suppose m is the largest element of the domain of g^i and $\ell > g^i(m)$ is the largest element of the domain of f. For some $t > f(\ell)$, we see $A_t(z) \neq A_{f\circ g^i(m)}(z)$. This change will be recorded in the change set, and $\sum d^i(B_{g^i(s)})$ will pay at most $d^i(z, m)$ for it.

We set $c(z, s) = d^i(z, m)$, and it is now safe to define $f(\ell + 1) \geq t$.

Another problem: our analysis has assumed that $g^i(m+1)$ is large.

In the first idea, we needed $g^i(m+1) > s$.

In the second idea, we needed $g^i(m+1) > \ell$.

g^i converges slower than f, so we can't guarantee that this will hold.

So we need another speed-up h^i.

Third idea:

We will build functions f and h^i. If $\langle A_s \rangle_{s\in\omega}$ is total, f will be total. If g^i is total, h^i will be total.

B is again the change set for $\left\langle A_{f(s)} \right\rangle_{s\in\omega}$. If g^i is total and $\sum d^i(A_{f\circ g^i(s)}) < \infty$, we will make $\sum d^i(B_{g^i\circ h^i(s)}) < \infty$.

At stage s, suppose m is the largest element of the domain of h^i and $\ell > g \circ h^i(m)$ is the largest element of the domain of f. For some $t > f(\ell)$, we see $A_t(z) \neq A_{f\circ g^i\circ h^i(m)}(z)$.

We set $c(z, s) = d^i(z, m)$, and it is now safe to define $f(\ell + 1) \geq t$ and $h^i(m + 1)$ such that $g^i \circ h^i(m + 1) > \ell$.

Another problem: c needs to be benign.

d^i need not be benign, so constantly setting $c = d^i$ will be a problem.

Our solution is to drip feed cost increases. Instead of setting $c(z,s) = d^i(z,m)$, we set $c(z,s) = 2^{-s}$, and then wait until $A_s(z)$ converges at some stage $s_1 > s$. While we wait, we do not define f.

If $A_s(z) = A_{f \circ g^i \circ h^i(m)}(z)$, z is no longer a problem. We can continue our definitions ignoring z.

Otherwise, we define $c(z, s_1) = 2 \cdot c(z,s)$, and then wait until $A_{s_1}(z)$ converges at some stage $s_2 > s_1$. If $A_{s_1}(z) = A_{f \circ g^i \circ h^i(m)}(z)$, z is no longer a problem. Otherwise, define $c(z, s_2) = 2 \cdot c(z, s_1)$.

If z remains a problem, we continue with $s_2 < s_3 < s_4 \ldots$ until $c(z, s_r) \geq d^i(z,m)$. Only then are we safe to continue the definition of f.

To be benign, we must bound the number of times c increases to 2^{-n}.

Except for the very first $c(z, s) = 2^{-s}$, we never increase c unless $\sum c_s(A_s) + \sum d^i(A_{f \circ g^i(s)})$ has committed to paying at least half the increase.

We assume that $\sum c_s(A_s) \leq 2^k$ and $\sum d^i(A_{f \circ g^i(s)}) \leq 1$. (We stop increasing c once we see these sums exceed their bounds.)

This lets us get a computable bound on the increases. □

Analysis of the construction shows that $\dot{c}$ is $2^{n^{O(1)}}$-benign.

So the class $\{X : X \text{ obeys all } 2^{n^{O(1)}}\text{-benign cost functions}\}$ is also a c.e. generated ideal.

It is strictly between SJT and K-trivial.

Recall that A is called a *base for* Q if there is an $X \in Q^A$ with $A \leq_T X$.

Theorem (Nies)

If A is a base for Demuth randomness, then A is SJT.

Theorem (Greenberg, T)

There is an SJT which is not a base for Demuth randomness.

There are three places to relativize Demuth randomness.

An A-Demuth test is a sequence $\langle V_n \rangle_{n \in \omega}$ of open sets with $\lambda(V_n) \leq 2^{-n}$ such that:

1. There is an A-computable function $f : \omega \times \omega \to \omega$;
2. There is an A-computable function $g : \omega \to \omega$ with $|\{s : f(n,s) \neq f(n,s+1)\}| \leq g(n)$; and
3. $\lim_s f(n,s)$ is a $\Sigma^0_1(A)$-index for V_n.

It can be shown that the third relativization gives no additional power.

Nies's proof only used the first relativization.

Greenberg & T's counterexample only used the second.

This leads to the following definition:

Definition

For an oracle A, an *A-Demuth$_{BLR}$ test* is a sequence $\langle V_n \rangle_{n\in\omega}$ of open sets with $\lambda(V_n) \leq 2^{-n}$ such that there is an A-computable function $f : \omega \times \omega \to \omega$ and a computable function $g : \omega \to \omega$ with $|\{s : f(n,s) \neq f(n,s+1)\}| \leq g(n)$ and $\lim_s f(n,s)$ a Σ^0_1-index for V_n.

$\emptyset$-Demuth$_{\mathrm{BLR}}$ tests are precisely Demuth tests.

Theorem (Bienvenu, Downey, Greenberg, Nies, T)

The SJT are precisely the bases for Demuth$_{BLR}$ randomness.

Definition

For a cost function c, a *c-test* is an effectively given sequence of nested Σ^0_1-class $\langle V_n \rangle_{n\in\omega}$ with $\lambda(V_n) \leq \lim_s c(n,s)$.

Lemma

If X is MLR and fails a c-test, then X computes every c.e. set which obeys c.

Proof.

Fix a c-test $\langle V_n \rangle$ capturing X and a c.e. enumeration of a set A obeying c. Assume that $\lambda(V_{n,s}) \leq c(n,s)$.
If $s \geq n$ and $n \notin A_s$, enumerate n into W^Z for every $Z \in V_{n,s}$.
If $s > n$ and $n \in A_s \setminus A_{s-1}$, enumerate $V_{n,s-1}$ into S.

The total measure enumerated into S is bounded by $\sum c(A_s)$, so S is a Solvay test. So X is only enumerated into S finitely many times, and thus W^X enumerates something with only finitely many differences from the complement of A. □

Lemma

Every c-test for a benign c is covered by a weak Demuth test, and conversely.

Proof.

Given a weak Demuth test $\langle V_n \rangle_{n\in\omega}$, with corresponding functions $f : \omega \times \omega \to \omega$ and $g : \omega \to \omega$, assume that $V_{f(n,s),s} \supseteq V_{f(n+1,s),s}$ at every stage.

Let $U_n = \bigcup_{s \geq n} V_{f(n,s),s}$. Define $c(n,s) = \lambda(U_{n,s})$. c is $g(n+1)$-benign.

Conversely, given a c-test $\langle V_n \rangle_{n\in\omega}$, let $\ell_0^n, \ell_1^n, \ldots, \ell_r^n$ be from the definition of benignness. Define $U_n = V_{\ell_r^n}$. □

So every MLR which is not weak Demuth random computes every SJT.

Question

Are the weak Demuth randoms precisely the MLR which do not compute every SJT?

We cannot hope to use the same proof as for K-trivial and OWR.

Theorem (Greenberg, T)

Every SJT is computable from a Δ^0_3 Demuth random.

Question

Can this be improved to Δ^0_2?

Thank you.